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Holography, diffeomorphisms, and scaling violations in the CMB

Finn Larsen and Robert McNees

Michigan Center for Theoretical Physics, University of Michigan
Ann Arbor, MI-48109, U.S.A.
E-mail: larsenf@umich.edu, ramcnees@umich.edu

Abstract: We analyze diffeomorphism invariance in inflationary spacetimes regulated by a boundary at late time. We present the action for quadratic fluctuations in the presence of a boundary, and verify that it is gauge invariant precisely when the correct local counterterms are included. The scaling behavior of bulk correlation functions at the boundary is determined by Callan-Symanzik equations which predict scaling violations in agreement with the standard inflationary predictions for spectral indices of the CMB.

Keywords: Renormalization Group, AdS-CFT and dS-CFT Correspondence, Cosmology of Theories beyond the SM, Physics of the Early Universe.

Contents

1. Introduction

It is useful to think of cosmological evolution in terms of a family of spatial slices, with time appearing as a parameter identifying the individual slices. The cosmological wave function [1] is then a functional which, in the Schrödinger picture, takes the schematic form:

$$\Psi[\phi] \sim e^{iS[\phi]} \tag{1.1}$$

where ϕ is a set of variables defined on a three dimensional equal-time slice. In the semi-classical approximation $S[\phi]$ can be identified as the Hamilton-Jacobi (H-J) functional, a versatile tool in cosmology [2]–[5]. The H-J functional is defined as the on-shell action, interpreted as a functional of the dynamical variables on the equal time slice; so the H-J form of the dynamical problem involves gravitational physics on a 'bulk' manifold which ends at a 'boundary', the equal time slice under consideration. It is therefore well suited to the study of gravitational physics on manifolds with a boundary, a problem which also has many other applications, such as in brane-world models.

In the present paper we study the gravity-scalar system on a manifold which, for definiteness, we take as an inflationary spacetime. The on-shell actions for these spacetimes contain late time divergences which can be regulated by truncating the manifold at a late time, resulting in a boundary. As a result the action in (1.1) is a functional of the 'boundary data', the variables ϕ evaluated on the spatial slice corresponding to the late time cut-off. Our main results are:

- Diffeomorphism Invariance is not automatic in the presence of such a boundary. The simplest way to preserve diffeomorphism invariance is to introduce local counterterms on the boundary. We determine their form.
- We compute the quadratic action for fluctuations around a manifold with a boundary. We present our result in terms of gauge invariant variables.
- We interpret the spectral indices of the Cosmic Microwave Background (CMB) in terms of scaling violations of a 'boundary theory'. This perspective is holographic in character, since a three-dimensional theory controls the four-dimensional physics.

The starting point for our discussion is the straightforward and explicit computation of the H-J functional [6, 7]. The result of this computation suffers from a divergence as the time of the slice is taken to future infinity. This divergence is dominated by large wavelengths, and so it can be cancelled by adding a local boundary term — a counterterm — to the action.

However, the naïve computation suffers from additional, and seemingly more serious, problems. As we will explain, the presence of an arbitrary boundary, introduced to regulate divergences, renders the H-J functional inconsistent with the full set of four dimensional diffemorphisms. This failure of local reparametrization invariance can be remedied by supplementing the standard action with a boundary term. We will show that this boundary term, designed to restore diffeomorphism invariance, is in fact the same as the counterterm needed to cancel infrared divergences.

The gravity side of the AdS/CFT correspondence [8]–[10] involves the on-shell action on a (possibly deformed) AdS-space. It is well known that this action exhibits infrared divergences due to the behavior of the metric near the boundary at spatial infinity. These divergences are naturally cancelled by the introduction of boundary counterterms [11]–[14] . In the context of the AdS/CFT correspondence, the counterterms are interpreted in the dual conformal field theory as the usual counterterms needed to cancel ultraviolet divergences in quantum field theory [12]; but their origin on the gravity side is less clear.

Although we work with cosmological spacetimes for definiteness, our results are valid for asymptotically AdS-spaces as well. This suggests a new perspective on the counterterms in AdS/CFT, which is rooted solidly in gravity: counterterms are needed to maintain diffeomorphism invariance.

In standard cosmological perturbation theory it is customary to implement diffeomorphism invariance by introducing gauge invariant physical observables [15]–[19]. We will extend this result to include the boundary, and present the quadratic action for fluctuations in this more general case, again written in gauge invariant form. This is one of our main results.

Diffeomophism invariance also constrains the dependence of the boundary theory on the gauge invariant variables. The origin of these additional constraints are the diffeomorphisms acting on the direction normal to the boundary. These transformations are implemented as scaling symmetries on the three-dimensional equal time surface, and so their effect is to determine the scale dependence of the correlation functions. We refer to the equations determining the scale dependence as the Callan-Symanzik equations.

The Callan-Symanzik equations can be solved using techniques that are standard from renormalization group theory [20]. The result of this analysis is general formulae for the correlation functions of the theory, determined by symmetries alone. To exploit these formulae, one must add dynamical input, e.g. from slow roll inflation. Given this input, our expressions are renormalization group improved versions of the more conventional results. We consider both scalar and tensor fluctuations and determine, in particular, the scalar and tensor mode spectral indices, n_s and n_t, which characterize the scale dependence of the CMB.

The terminology introduced to describe consequences of diffeomorphism invariance — counterterm, Callan-Symanzik equations, and the Ward identity — is that of a local quantum field theory on the equal time slice. In the context of the AdS/CFT correspondence our terminology fully justified but, in cosmology, it refers to a conjectured dS/CFT correspondence [21]–[26]. Such ideas, though rather speculative, have been used to address inflation [6, 7, 27, 28]. It would be extremely interesting if a truly holographic theory of cosmology could be established. However, whether cosmological holography is true or not, the counterterms we discuss are a universal part of the gravitational action, determined from diffeomorphism invariance alone. It may be that gravity is characterized by "infrared universality classes" which would be similar to the "ultraviolet universality classes" familiar from quantum field theory. In quantum field theory truly short distances decouple from long distance physics and similarly it seems that, in cosmology, truly large distance physics, beyond the horizon, decouples from short distance physics, i.e. observable cosmology [29]. The notion of "infrared universality classes" could develop into a framework for addressing the notorious fine-tuning problems in cosmology. This might apply not only to the fine-tuning problems normally associated with inflation, but also to other naturalness problems associated with the decoupling of long and short distance physics, such as the cosmological constant problem.

This paper is organized as follows. In section 2 we review the cancelling of infrared divergences *via* the introduction of counterterms. We then present an argument that identifies the origin of the counterterms as diffeomorphism invariance. In section 3 we compute the quadratic action of fluctuations around the background. To do this, we review the standard notion of gauge invariance in the bulk theory, and show how this can be extended to the boundary, precisely when the correct boundary terms are introduced. In section 4 we discuss the consequences of diffeomorphism invariance for the form of correlation function. The constraints are summarized by a master equations which, in a special case, reduces to the Callan-Symanzik equation. In section 5 we solve the Callan-Symanzik equation. In particular we determine the spectral parameters of cosmological inflation as the scaling violations of the theory. Finally, in section 6, we conclude with an outlook for further developments.

2. Counterterms on the boundary

In this section we review the appearance of infrared divergences in the gravitational action and their cancellation by counterterms on the regulating boundary. We then discuss how the presence of the boundary introduces sources in the equations of motion and, by a related mechanism, violates diffeomorphism invariance. This shows that counterterms are needed to preserve a crucial symmetry, diffeomorphism invariance.

2.1 The setting

The simplest and most common setting for discussing inflation is four-dimensional Einstein gravity coupled to a single scalar field. For a review of scalar field inflation, see [30]–[33] and references therein. The action of the theory is:

$$S = \int_{\mathcal{M}} d^4x\sqrt{g}\left(\frac{1}{16\pi G}\,R - \frac{1}{2}\,\nabla^\mu\varphi\nabla_\mu\varphi - V(\varphi)\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\sqrt{\tilde{g}}\,K\,. \tag{2.1}$$

The cosmological spacetime $\mathcal{M}$ will have a spacelike boundary $\partial\mathcal{M}$ defined by a timelike unit normal n^μ. The metric on $\mathcal{M}$ is $g_{\mu\nu}$ and the induced metric on $\partial\mathcal{M}$ is $\tilde{g}_{\mu\nu}$. The Gibbons-Hawking term [34] ensures that the action poses a well defined variational problem. In the rest of this paper we use units where $8\pi G = 1$.

During the inflationary epoch the metric and the scalar field are approximately spatially homogeneous. They take the form:

$$\begin{aligned} \varphi(\vec{x},\tau) &= \varphi(\tau) + \chi(\vec{x},\tau) \\ g_{\mu\nu}(\vec{x},\tau) &= a(\tau)^2\,\eta_{\mu\nu} + h_{\mu\nu}(\vec{x},\tau) \end{aligned} \tag{2.2}$$

where $\eta_{\mu\nu} = \mathrm{diag}\{-1,1,1,1\}$ is the usual Minkowski metric. The $\vec{x}$ are spatial coordinates and τ is conformal time, which runs over $\tau \in (-\infty, 0)$. Standard slow-roll inflation assumes that φ is approximately constant, corresponding to a background that is approximately de Sitter space, i.e. $a(\tau) \propto \tau^{-1}$. These 'quasi-de Sitter' spacetimes will be our primary example but most of our results in fact apply to general FRW cosmologies.

We refer to the spatially homogenous parts of (2.2) as the "background". The equations of motion governing the background are the standard FRW equations:

$$\varphi'' + 2\mathcal{H}\varphi' + a^2\partial_\varphi V = 0 \tag{2.3}$$

$$3\left(\frac{\mathcal{H}}{a}\right)^2 = \frac{1}{2}\left(\frac{\varphi'}{a}\right)^2 + V\,. \tag{2.4}$$

A prime denotes a derivative with respect to the conformal time τ, and $\mathcal{H} = a'/a$ is the conformal Hubble factor.

The dynamics of the small, spatially inhomogenous fluctuations in (2.2) is described by the action (2.1), evaluated as a series in χ and $h_{\mu\nu}$ around the background. Terms in the action which are linear in the fluctuations are "first order", and quadratic terms are "second order". According to inflation [35]–[42] these fluctuations are responsible for the minute variations in the CMB that we observe today [43, 44, 45].

2.2 Counterterms: cancelling divergences

A powerful way to analyze a dynamical system is the Hamilton-Jacobi (H-J) formalism. The linchpin of this formalism is the H-J functional, defined as the on-shell action:

$$S_{\mathrm{H-J}}[\varphi, \tilde{g}_{\mu\nu}] = S_{\text{on shell}}[\varphi, \tilde{g}_{\mu\nu}]\,. \tag{2.5}$$

The dynamical variables φ and $\tilde{g}_{\mu\nu}$ are defined on a spatial slice parametrized by the conformal time τ. The initial conditions are usually left implicit but, in the cosmological context, it is natural to specify them by imposing regularity as $\tau \to -\infty$. The H-J functional can be interpreted in the semiclassical regime as the phase of the cosmological wave function (1.1). The regularity conditions at early times then specifies the initial conditions as the Hartle-Hawking state.

The H-J functional corresponding to the action (2.1) diverges as the time parameter is taken to future infinity $\tau \to 0^-$. This can be seen in an elementary way by making simplifying assumptions about the background and the fluctuations. Consider, for example, a homogeneous massless scalar field in the background of pure de Sitter space. The bulk equations of motion then determine the scalar field as a linear combination of decreasing and increasing solutions, with the regularity condition at early times excluding the "decreasing" solution. However, the on-shell action, evaluated on the "increasing" solution, diverges at late times. As a result the H-J functional suffers an infrared divergence (for details see [7]). In this elementary derivation the origin of the infrared divergence is that, in general, no solution gives regular on-shell actions at both early and late times.

The divergences can be characterized much more generally by considering the hamiltonian constraint which, in the present context, is implemented by the H-J equation:

$$\left(\frac{1}{\sqrt{\tilde{g}}}\tilde{g}_{\mu\nu}\frac{\delta S}{\delta \tilde{g}_{\mu\nu}}\right)^2 - 2\left(\frac{1}{\sqrt{\tilde{g}}}\frac{\delta S}{\delta \tilde{g}_{\mu\nu}}\right)\left(\frac{1}{\sqrt{\tilde{g}}}\frac{\delta S}{\delta \tilde{g}^{\mu\nu}}\right) = V - \frac{1}{2}\mathcal{R} + \frac{1}{2}\vec{D}\varphi\cdot\vec{D}\varphi\,. \tag{2.6}$$

The dominant terms in the solutions are captured by the inverse metric expansion:

$$S = \int_{\partial\mathcal{M}} d^3x\sqrt{\tilde{g}}\left(U(\varphi) + M(\varphi)\vec{D}\varphi\cdot\vec{D}\varphi + C(\varphi)\tilde{R} + \cdots\right) \tag{2.7}$$

In this form the divergence of the H-J functional is a consequence of the divergence of the FRW scale factor. The terms displayed explicitly in the *ansatz* (2.7) are sufficient to capture *all* divergences in the action, even when the terms indicated by dots are neglected. This characterization of the divergences in the action is rather general because it applies to the full cosmology, with inhomogeneous fluctuations, and it makes only modest assumptions about the background.[1] It should also be noted that this type of local ansatz for the on-shell action is commonly used in the study of holographic renormalization group flows in AdS/CFT [46]–[52].

Inserting the *ansatz* (2.7) in the H-J equation, and solving order-by-order in the inverse metric expansion, we find differential equations that determine the functions $U(\varphi)$, $M(\varphi)$, and $C(\varphi)$:

$$0 = \frac{3}{4}U^2 - \frac{1}{2}(\partial_\varphi U)^2 - V \tag{2.8}$$
$$0 = 1 + U\,C - 2\,\partial_\varphi U\,\partial_\varphi C \tag{2.9}$$
$$0 = 1 - U\,M - 4\partial_\varphi U\,\partial_\varphi C + 2\,\partial_\varphi U\,\partial_\varphi M + 4\partial_\varphi^2\,U\,M\,. \tag{2.10}$$

The first equation (2.8) reproduces the Friedmann equation (2.4) for a homogenous background if we identify $U(\varphi)$ and its first derivative as:

$$U(\varphi) = -2\,\frac{\mathcal{H}(\varphi)}{a(\varphi)} \qquad \partial_\varphi U = \frac{\varphi\,'}{a}\,. \tag{2.11}$$

It is important to note that the equation (2.8) and the conditions (2.11) specify the functional dependence of $U(\varphi)$ on the field φ, and not just the value that $U(\varphi)$ and its first derivative take. It may therefore be used as the leading counterterm, even for backgrounds which contain spatial inhomogeneities.

At this point we have shown that the H-J functional S diverges, and that the divergences are characterized in general by (2.7). We can use this information to isolate the finite part of the H-J functional as follows: first, regulate the divergences that appear in the action by cutting the spacetime off at some late time τ_0. Then introduce the following counterterms, intrinsic to the spatial hypersurface defined by the condition $\tau = \tau_0$.

$$S_{\rm CT}(\tau_0) = \int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\left(U(\varphi) + M(\varphi)\vec{D}\varphi\cdot\vec{D}\varphi + C(\varphi)\tilde{R}\right) \tag{2.12}$$

The 'total action' $S_{\rm tot}$ is then given by the original action, minus the contribution of the counterterms on the regulating boundary:

$$S_{\rm tot}(\tau_0) = S(\tau_0) - S_{\rm CT}(\tau_0) \tag{2.13}$$

This action $S_{\rm tot}$ is finite as $\tau_0 \to 0$. A simple physical interpretation of this procedure is that S is the action of the complete cosmology, $S_{\rm CT}$ captures the divergences present in the background, and their difference, $S_{\rm tot}$, represents the action of the fluctuations alone. This interpretation should not be taken too seriously: since $S_{\rm CT}$ is a true functional of the

[1] The conformal factor must diverge at late times. This is satisfied for all expanding cosmologies. In addition, we use the de Sitter form of the metric to estimate when the series (2.7) can be truncated. This is satisfied for "quasi-de-Sitter spacetimes", such as those that appear in inflationary cosmology.

spacetime and so, when expanding around a background, there will be terms attributable to the fluctuations. A more appropriate, albeit more abstract, terminology is that of standard quantum field theory: τ_0 is the *physical cut-off*, (2.12) are the local *counterterms*, and (2.13) is the *renormalized action*.

2.3 Sources at the boundary

The counterterms are intrinsic to the regulating boundary, so they do not affect the *bulk* equations of motion derived from (2.1). However, we need to consider the equations of motion *on* the boundary. The equations of motion are determined by the variational principle. The first order variation of the original action (2.1) is:

$$\delta S = \int_{\mathcal{M}} d^4x\,\sqrt{g}\left(\frac{1}{2}h^{\mu\nu}\left(T_{\mu\nu} - G_{\mu\nu}\right) + \chi\left(\nabla^2\varphi - \partial_\varphi V\right)\right) + \int_{\partial\mathcal{M}} d^3x\sqrt{\tilde{g}}(h^{\mu\nu}\,\pi_{\mu\nu} + \chi\,\pi_\varphi). \tag{2.14}$$

In the usual variational principle we consider only variations of the fields that vanish on the boundary $\partial\mathcal{M}$; and so we discard the boundary term in (2.14). Extremizing the action then gives the Einstein equations $G_{\mu\nu} = T_{\mu\nu}$ and the scalar equation of motion $\nabla^2\varphi = \partial_\varphi V$, as expected.

The equations of motion *on* the regulating boundary with conformal time $\tau = \tau_0$ are determined by keeping the variations $h_{\mu\nu}$ and χ *arbitrary* there. We see that:

$$\pi_\varphi = \frac{1}{\sqrt{\tilde{g}}}\frac{\delta S}{\delta\varphi} \qquad \pi^{\mu\nu} = \frac{1}{\sqrt{\tilde{g}}}\frac{\delta S}{\delta\tilde{g}_{\mu\nu}} \tag{2.15}$$

correspond to sources on the boundary which act as net forces. The sources $\pi_{\mu\nu}$ and π_φ are the canonical momenta which, in general, are non-vanishing. The action is not extremized, $\delta S \neq 0$, unless these terms are cancelled by external forces.

In the present context the appropriate action is in fact $S_{\rm tot} = S - S_{\rm CT}$, rather than just S, and the boundary counterterms provide an external force. The momenta due to the counterterms are:

$$P_\varphi = \frac{1}{\sqrt{\tilde{g}}}\frac{\delta S_{CT}}{\delta\varphi} \qquad P^{\mu\nu} = \frac{1}{\sqrt{\tilde{g}}}\frac{\delta S_{CT}}{\delta\tilde{g}_{\mu\nu}} \tag{2.16}$$

so the conditions for the variation of $S_{\rm tot}$ to vanish on the boundary are:

$$\pi_\varphi = P_\varphi \qquad \pi^{\mu\nu} = P^{\mu\nu} \tag{2.17}$$

It is not easy to solve the equations (2.17) in general. In the special case of a homogeneous background it is straightforward to compute the canonical momenta in terms of the scale factor $a(\tau)$ and its derivatives, and then find the counterterm momenta with appropriate variations. The result is:

$$U(\varphi) = -2\,\frac{\mathcal{H}(\varphi)}{a(\varphi)} \qquad \partial_\varphi U = \frac{\varphi'}{a}\,. \tag{2.18}$$

This agrees precisely with the result (2.11) for the counterterms found by expanding the H-J equation and identifying divergent terms.

In some contexts it is a consistency requirement that the equations of motion are satisfied, even on the boundary. Thus it is possible to reverse the logic and take the equations of motion in the boundary as the starting point which motivates the introduction of the counterterms in the first place. From this point of view the infrared divergences and, in particular, the limit $\tau_0 \to 0^-$ play a secondary role. However, we will not pursue this in detail, but rather go one step further and motivate the counterterms from symmetries that hold even when the equations of motion do not. This is the subject we turn to next.

2.4 Counterterms: diffeomorphism invariance

Perhaps the best way to motivate boundary counterterms is that they are required to maintain diffeomorphism invariance on spacetimes with a boundary. This subsection presents the argument.

Diffeomorphism invariance states that the physics of a system is independent of the coordinate system chosen to describe it. Stated as a local symmetry, it means that the action S of a theory containing gravity is invariant under the infinitesimal local change of coordinates $x^\mu \to x^\mu + \epsilon^\mu(x)$. Under such a coordinate change the metric transforms as:

$$g_{\mu\nu} \to g_{\mu\nu} - \nabla_\mu \epsilon_\nu - \nabla_\nu \epsilon_\mu \tag{2.19}$$

and the scalar φ transforms as:

$$\varphi \to \varphi - \epsilon^\mu \nabla_\mu \varphi \tag{2.20}$$

The variation of the action (2.1) under an infinitesimal diffeomorphism is:

$$\begin{aligned} \delta_\epsilon S = & \int_{\mathcal{M}} d^4x \sqrt{g} \left[\nabla^\mu \epsilon^\nu \left(G_{\mu\nu} - T_{\mu\nu} \right) - \epsilon^\mu \nabla_\mu \varphi \left(\nabla^2 \varphi - \partial_\varphi V \right) \right] + \\ & + \int_{\partial\mathcal{M}} d^3x \sqrt{\tilde{g}} \left[-2\pi_{\mu\nu} \, \nabla^\mu \epsilon^\nu - \pi_\varphi \, \epsilon^\mu \nabla_\mu \varphi \right] \end{aligned} \tag{2.21}$$

After integration by parts this becomes:

$$\begin{aligned} \delta_\epsilon S = & \int_{\mathcal{M}} d^4x \sqrt{g} \left[-\epsilon^\nu \nabla^\mu G_{\mu\nu} + \epsilon^\nu \left(\nabla^\mu T_{\mu\nu} - \nabla_\nu \varphi \left(\nabla^2 \varphi - \partial_\varphi V \right) \right) \right] + \\ & + \int_{\partial\mathcal{M}} d^3x \sqrt{\tilde{g}} \left[n^\mu \epsilon^\nu \left(G_{\mu\nu} - T_{\mu\nu} \right) - 2\pi_{\mu\nu} \, \nabla^\mu \epsilon^\nu - \pi_\varphi \, \epsilon^\mu \nabla_\mu \varphi \right]. \end{aligned} \tag{2.22}$$

The contracted Bianchi identity implies $\nabla^\mu G_{\mu\nu} = 0$ and the remaining bulk terms vanish as well.[2] The fact that *all* bulk terms vanish identically is independent of the equations of motion, which have not been imposed.

Variations of the action under diffeomorphisms are thus captured by certain boundary terms. These terms can be written in a more illuminating form by splitting the vector ϵ^μ into its normal and tangential components:

$$\epsilon^\mu = -n^\mu n_\nu \epsilon^\nu + \tilde{g}^\mu{}_\nu \, \epsilon^\nu \,. \tag{2.23}$$

[2]To verify that $\nabla^\mu T_{\mu\nu} = \nabla_\mu \varphi (\nabla^2 \varphi - \partial_\varphi V)$ simply insert the explicit form of the energy-momentum tensor $T_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - g_{\mu\nu} (\frac{1}{2} (\nabla \varphi)^2 + V(\varphi))$.

It can be shown that the Gauss-Codazzi equations and the definitions of canonical momenta imply:

$$\begin{aligned} n^\mu n^\nu (T_{\mu\nu} - G_{\mu\nu}) &= 2\pi^{ij}\pi_{ij} - \pi^i_i \pi^j_j - \frac{1}{2}\mathcal{R} + \frac{1}{2}\pi_\varphi^2 + \frac{1}{2}\vec{D}\varphi\cdot\vec{D}\varphi + V = \mathscr{H} \\ n^\mu \epsilon^i (G_{\mu i} - T_{\mu i}) &= \pi_\varphi \; \epsilon^i D_i\varphi - 2\epsilon^i D^j \pi_{ij} = -\epsilon^i \mathscr{H}_i \end{aligned} \tag{2.24}$$

and so we can write (2.22) as:

$$\delta_\epsilon S = \int_{\partial\mathcal{M}} d^3x \sqrt{\tilde{g}} \left[n_\lambda \epsilon^\lambda \, \mathscr{H} - \epsilon^i \mathscr{H}_i - 2\pi^{\mu\nu}\nabla_\mu \epsilon_\nu - \pi_\varphi \epsilon^\mu \nabla_\mu \varphi \right] \tag{2.25}$$

This makes it manifest that the only components of the equations of motion $G_{\mu\nu} - T_{\mu\nu} = 0$ which appear as generators of the diffeomorphisms are those proportional to the contraints $\mathscr{H} = \mathscr{H}_i = 0$.

Splitting the remaining terms into normal and tangential components gives:

$$2\pi^{\mu\nu}\nabla_\mu\epsilon_\nu = 2\pi^{ij} D_i \epsilon_j + 4\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^i_i\pi^j_j\right) n_\lambda \epsilon^\lambda \tag{2.26}$$

$$\pi_\varphi \; \epsilon^\mu \nabla_\mu \varphi = \pi_\varphi \epsilon^i D_i \varphi + \pi_\varphi^2 \; n_\lambda \epsilon^\lambda \tag{2.27}$$

and the complete expression can be simplified as:

$$\delta_\epsilon S = -\int_{\partial\mathcal{M}} d^3x \sqrt{\tilde{g}} \, n_\lambda \epsilon^\lambda \, \mathscr{L} \tag{2.28}$$

In other words, the lagrangian density transforms as a scalar field (2.20), as it should.[3] The result (2.28) shows that the action is invariant under reparameterizations of the spatial coordinates but, when there is a boundary present, the action is *not* invariant under reparametrizations of the direction normal to the boundary. One often neglects this variation by assuming that the coordinate transformations are localized in the bulk, so that the normal component of ϵ^μ falls off sufficiently rapidly near the boundary $\partial\mathcal{M}$. In the present context we are not entitled to ignore this term. Indeed it is of great interest, because it characterizes the violation of diffeomorphism invariance due to the non-covariant regulator we have introduced.

So far we have neglected the counterterms in the discussion of diffeomorphism invariance. The variation of the renormalized action $S_{\text{tot}} = S - S_{\text{CT}}$ under an infinitesimal diffeomorphism follows from (2.25) and the definitions (2.16). It can be written as:

$$\delta_\epsilon S_{\text{tot}} = \int_{\partial\mathcal{M}} d^3x \sqrt{\tilde{g}} \left[n_\lambda \epsilon^\lambda \, \mathscr{H} - \epsilon^i \mathscr{H}_i - 2\left(\pi^{\mu\nu} - P^{\mu\nu}\right)\nabla_\mu\epsilon_\nu - \left(\pi_\varphi - P_\varphi\right)\epsilon^\mu\nabla_\mu\varphi \right] \tag{2.29}$$

Invariance under the full four-dimensional diffeomorphisms requires that this expression vanishes. It is natural to impose the usual hamiltonian and momentum constraints:

$$\mathscr{H} = 0 \qquad \mathscr{H}_i = 0\,. \tag{2.30}$$

[3] The lagrangian $\mathscr{L}$ that appears here is in first order form, i.e. the Gibbons-Hawking term has been cancelled through integration by parts.

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The remaining terms in (2.29) are then similar to the boundary sources for momentum, discussed in subsection 2.3. However, here we have not used the equations of motion; so these terms constitute genuine violations of diffeomorphism invariance.

The point, of course, is that the counterterms, when appropriately chosen, can restore diffeomorphism invariance; indeed, this might be the most persuasive motivation for introducing the counterterms in the first place. The condition for diffeomorphism invariance in the presence of a boundary is:

$$\pi_{\mu\nu} = P_{\mu\nu} \qquad \pi_\varphi = P_\varphi \tag{2.31}$$

as in subsection 2.3. As explained there, these equations can be integrated, for homogeneous backgrounds, to determine the counter-terms as:

$$U(\varphi) = -2\frac{\mathcal{H}(\varphi)}{a(\varphi)} \qquad \partial_\varphi U = \frac{\varphi'}{a} \tag{2.32}$$

Note that the equalities in (2.31) relate the values that the momenta take at the boundary, and not their functional form. This can be clearly seen when one recalls that the canonical momenta π_{ij} and π_φ depend on the normal derivatives of the boundary data, whereas the counterterms, and hence their contribution to the momenta, are entirely intrinsic to the boundary.

In section 3 we will consider inhomogeneous backgrounds as well, and show similar agreements for the subleading counterterms $M(\varphi)$ and $C(\varphi)$. Diffeomorphism invariance is therefore equivalent to the condition that divergences cancel at late times.

3. Gauge invariance and the quadratic action

The purpose of this section is to derive the quadratic action for fluctuations in the presence of a boundary, and demonstrate that it is gauge invariant. We first review gauge invariance for bulk fluctuations. We then include boundaries, and present the action in this case. We show that this action can be written in terms of gauge invariant variables precisely when the correct counterterms have been included.

3.1 Gauge invariance for inflationary spacetimes

We consider a spatially homogenous background defined by a scalar field $\varphi(\tau)$ and a scale factor $a(\tau)$. The fluctuations correspond to the terms χ and $h_{\mu\nu}$ appearing in (2.2), with $h_{\mu\nu}$ parameterized by:[4]

$$h_{\mu\nu} = a(\tau)^2 \begin{pmatrix} 2\,\Phi(\vec{x},\tau) & \partial_i B(\vec{x},\tau) \\ \partial_i B(\vec{x},\tau) & \phi_{ij}(\vec{x},\tau) + 2(\psi(\vec{x},\tau)\,\delta_{ij} - \partial_i\partial_j E(\vec{x},\tau)) \end{pmatrix} \tag{3.1}$$

The ϕ_{ij} are tensor modes (with respect to a three-dimensional spatial hypersurface) which are traceless and transverse:

$$\delta_{ij}\,\phi_{ij} = 0 \qquad \partial_i\phi_{ij} = 0\,. \tag{3.2}$$

[4]We consider only scalar and tensor fluctuations in the metric. Vector fluctuations do not couple to energy density perturbations in the cases we are interested in.

The remaining 5 fields Φ, ψ, B, E, χ are scalars. Two of them have familiar interpretations: the function Φ, in the newtonian limit, is the gravitational potential. The function ψ is known as the "curvature perturbation" and is related to the intrinsic curvature $\tilde{R}$ on a constant τ hypersurface by:

$$\delta\tilde{R} = -\frac{4}{a(\tau)^2}\,\vec{\partial}^2\psi \tag{3.3}$$

Diffeomorphism invariance implies that the physical content of the fluctuations is limited to excitations which cannot be absorbed in a local change of coordinates. The tensor modes ϕ_{ij}, which are invariant under such a change of coordinates, contain two physical degrees of freedom corresponding to the two independent polarizations of gravitational waves. In addition there is a single physical scalar degree of freedom which is represented by the five scalars Φ, ψ, B, E, χ. Two of these fields can be eliminated by gauge conditions on the coordinates, and two more will be removed by constraints, leaving one physical scalar.

Unlike the tensor modes, the five scalars transform under a local change of coordinates. We are interested in identifying the gauge-invariant combinations of these fields that are related to measurable quantities. We can find the transformation properties of the scalars by writing out the transformation (2.19) of the metric, using the parametrization (3.1), and the scalar field under an infinitesimal diffeomorphism $x^\mu \to x^\mu + \epsilon^\mu$. This leads to the following transformation rules for the scalar perturbations:

$$\Phi \to \Phi + \mathcal{H}\delta\tau + (\delta\tau)' \tag{3.4}$$
$$\psi \to \psi - \mathcal{H}\delta\tau \tag{3.5}$$
$$B \to B - \varepsilon' + \delta\tau \tag{3.6}$$
$$E \to E + \varepsilon \tag{3.7}$$
$$\chi \to \chi - \varphi'\,\delta\tau \tag{3.8}$$

where $\epsilon^\tau = \delta\tau$ and $\epsilon^i = \partial_i\varepsilon$. It is straightforward to use these transformation rules to identify gauge-invariant combinations of the scalar fluctuations. We will illustrate this by considering four common gauge choices, each with a clear physical interpretation. The coordinate transformations required to implement these gauge choices will help to identify a number of gauge-invariant quantities.

3.1.1 Comoving gauge

Comoving gauge (or total matter gauge) is defined by the condition that an observer sees no local flux of energy-momentum due to the fluctuations in the fields. In other words, it is defined by $T_{\tau i} = 0$. For the spatially homogenous background the zeroth order part of $T_{\tau i}$ vanishes. The first order fluctuation in $T_{\tau i}$ is given by:

$$\delta T_{\tau i} = \varphi'\,\partial_i\chi - a(\tau)^2\,\partial_i B\left(-\frac{1}{2}\left(\frac{\varphi'}{a}\right)^2 + V(\varphi)\right). \tag{3.9}$$

In comoving gauge this should vanish. Starting in an arbitrary gauge we can impose the condition $\delta T_{i\tau} = 0$ by making a coordinate transformation with:

$$\delta\tau = \frac{\chi}{\varphi'} \qquad \partial_\tau \epsilon = B + \frac{\chi}{\varphi'} \tag{3.10}$$

so that:

$$\chi_{\rm com} = \chi - \varphi' \delta\tau = 0 \tag{3.11}$$
$$B_{\rm com} = B - \partial_\tau \varepsilon + \delta\tau = 0\,. \tag{3.12}$$

In this gauge ψ becomes

$$\psi_{\rm com} = \psi - \frac{\mathcal{H}}{\varphi'}\chi\,. \tag{3.13}$$

Using the transformation rules (3.5) and (3.8) it is easy to verify that the variable ψ_{com}, the 'comoving curvature perturbation', is itself gauge invariant. Comoving gauge can be thought of as the coordinate choice in which the curvature perturbation ψ is equal to the gauge invariant quantity $\psi_{\rm com}$.

3.1.2 Flat gauge

Flat gauge is defined by $\psi_{\rm flat} = E_{\rm flat} = 0$. In this gauge the perturbation $\delta\tilde{R}$ of the spatial curvature vanishes.[5] To reach flat gauge from a generic coordinate system transform ψ and E as:

$$\psi_{\rm flat} = \psi - \mathcal{H}\delta\tau = 0 \tag{3.14}$$
$$E_{\rm flat} = E + \varepsilon = 0 \tag{3.15}$$

i.e. take $\delta\tau = \psi/\mathcal{H}$ and $\varepsilon = -E$. In flat gauge the observer does see a local flux in energy-momentum. It is due to a non-vanishing fluctuation in the scalar field, given by:

$$\chi_{\rm flat} = \chi - \frac{\varphi'}{\mathcal{H}}\psi\,. \tag{3.16}$$

This variable is gauge invariant and is related to the comoving curvature perturbation via the relation:

$$\chi_{\rm flat} = -\frac{\varphi'}{\mathcal{H}}\psi_{\rm com} \tag{3.17}$$

Except for a factor of $a(\tau)$ this is the same as Mukhanov's variable υ [17], which is given by:

$$\upsilon = a\,\chi_{\rm flat} = -a\,\frac{\varphi'}{\mathcal{H}}\psi_{\rm com} = -a\,\frac{\varphi'}{\mathcal{H}}\left(\psi - \frac{\mathcal{H}}{\varphi'}\chi\right). \tag{3.18}$$

Mukhanov's variable is often used to parameterize the scalar degree of freedom in inflation because its equation of motion is especially simple.

[5] The condition (3.3) as presented here only requires that $\psi = 0$ for $\delta\tilde{R}$ to vanish. More generally, when the background involves a non-zero spatial curvature, the variation in the intrinsic curvature of a constant-time hypersurface will also depend on E.

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3.1.3 Longitudinal gauge

The longitudinal (or conformal newtonian) gauge is defined by a diagonal metric, i.e. the off-diagonal components E and B vanish in longitudinal gauge. Starting from an arbitrary gauge this condition determines the gauge parameters as $\epsilon = -E$ and $\delta\tau = -(B + E')$. The non-trivial potentials become:

$$\Phi_B = \Phi - \mathcal{H}(B + E') - B' - E'' \tag{3.19}$$

$$\Psi_B = \psi + \mathcal{H}(B + E') \tag{3.20}$$

$$\chi_B = \chi + \varphi'(B + E') \,. \tag{3.21}$$

The metric functions Φ_B and Ψ_B in longitudinal gauge are the famous Bardeen variables [16].

3.1.4 Uniform energy density gauge

Another physically interesting gauge is the gauge in which an oberserver sees no local variation in the energy density ρ. Since ρ is a scalar the perturbation $\delta\rho$ transforms as:

$$\delta\rho \to \delta\rho - \rho'\,\delta\tau \tag{3.22}$$

so $\delta\rho_{\rm uni} = 0$ is reached by choosing $\delta\tau = \delta\rho/\rho'$. The curvature perturbation transforms as:

$$\psi_{\rm uni} = \psi - \frac{\mathcal{H}}{\rho'}\,\delta\rho \,. \tag{3.23}$$

As before, this combination of variables is invariant under a gauge transformation, and we can think of the Uniform Energy Density gauge as the coordinate system in which the curvature perturbation ψ is equal to the gauge-invariant quantity $\psi_{\rm uni}$.

We will not need $\psi_{\rm uni}$ in this paper and include it here only for completeness. However, we should point out that on super-horizon scales $\psi_{\rm uni}$ is essentially equal to $\psi_{\rm com}$. By computing $\delta\rho$ we can compare $\psi_{\rm uni}$ and $\psi_{\rm com}$. The energy density appears on the right hand side of the Friedmann equation (2.4). Its derivative is given by:

$$\rho' = -3\,\mathcal{H}\left(\frac{\varphi'}{a}\right)^2 \tag{3.24}$$

The variation in the energy density is given by:

$$\delta\rho = -3\frac{\mathcal{H}}{a^2}\,\varphi'\,\chi - \frac{1}{a^2}\,\vec{\partial}^2\psi \tag{3.25}$$

Using these expressions and the definitions of $\psi_{\rm uni}$ and $\psi_{\rm com}$ gives:

$$\psi_{\rm uni} = \psi_{\rm com} - \frac{1}{3(\varphi')^2}\,\vec{\partial}^2\psi \tag{3.26}$$

On super-horizon scales, where $\vec{k}^2$ is small compared to the Hubble scale, these variables are almost identical.

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3.2 The quadratic action in gauge-invariant variables

We now discuss the on-shell action to quadratic order in the fluctuations around the background. The total action, including the contributions from the counterterms, is given by:

$$\begin{aligned} S_{\text{tot}} &= \int_{\mathcal{M}_0} d^4x\sqrt{g}\left[\frac{1}{2}R - \frac{1}{2}\nabla^\mu\varphi\,\nabla_\mu\varphi - V(\varphi)\right] - \int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\,K - \\ &\quad - \int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\left[U(\varphi) + M(\varphi)\,\vec{D}\varphi\cdot\vec{D}\varphi + C(\varphi)\,\tilde{R}\right] \end{aligned} \tag{3.27}$$

The first order variation of the action is:

$$\begin{aligned} \delta S_{\text{tot}} &= \int_{\mathcal{M}_0} d^4x\sqrt{g}\left[\frac{1}{2}h^{\mu\nu}\left(T_{\mu\nu} - G_{\mu\nu}\right) + \chi\left(\nabla^2\varphi - \partial_\varphi V\right)\right] + \\ &\quad + \int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\left[h_{\mu\nu}\left(\pi^{\mu\nu} - P^{\mu\nu}\right) + \chi\left(\pi_\varphi - P_\varphi\right)\right]. \end{aligned} \tag{3.28}$$

The on-shell quadratic action is then obtained from the variation of the first order terms, neglecting terms that vanish because they are proportional to the background equations of motion. This gives:

$$\begin{aligned} \delta^2 S_{\text{tot}} &= \int_{\mathcal{M}_0} d^4x\sqrt{g}\left[\frac{1}{2}h^{\mu\nu}\,\delta\left(T_{\mu\nu} - G_{\mu\nu}\right) + \chi\,\delta\left(\nabla^2\varphi - \partial_\varphi V\right)\right] + \\ &\quad + \int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\left[h_{\mu\nu}\,\delta\left(\pi^{\mu\nu} - P^{\mu\nu}\right) + \chi\,\delta\left(\pi_\varphi - P_\varphi\right)\right]. \end{aligned} \tag{3.29}$$

Assuming a particular gauge before evaluating the variations in this expression would considerably simplify the calculation. However, choosing a gauge *assumes* gauge invariance, which we would like to understand explicitly. We therefore evaluate the action without choosing a gauge:

$$\begin{aligned} \delta^2 S_{\text{tot}} = \int_{\mathcal{M}_0} d^4x\sqrt{g}\frac{1}{a^2}\Big[& 2\,\Phi\,\vec{\partial}^2(\psi + \mathcal{H}(B + E')) - 4\,\mathcal{H}\,\Phi\left(\psi' + \mathcal{H}\,\Phi + \frac{1}{2}\varphi'\chi\right) - \\ & - 2\,\mathcal{H}\,\Phi\,\psi' - 2\,\mathcal{H}'\,\Phi^2 + \varphi'\,\Phi\,\chi' - \varphi''\,\Phi\,\chi - \\ & - 2\,\partial_i B\,\partial_i\left(\psi' + \mathcal{H}\,\Phi + \frac{1}{2}\varphi'\chi\right) + \\ & + 2\,(3\psi - \vec{\partial}^2 E)\,(\partial_\tau + 2\mathcal{H})\left(\psi' + \mathcal{H}\,\Phi + \frac{1}{2}\varphi'\chi\right) - \\ & - 2\psi\,\vec{\partial}^2\left(\psi - \Phi + B' + E'' + 2\,\mathcal{H}\left(B + E'\right)\right) - \\ & - \chi\,\chi'' - 2\,\mathcal{H}\,\chi\,\chi' - a^2\,\partial_\varphi^2 V\,\chi^2 + \chi\,\vec{\partial}^2\left(\chi + \varphi'\,(B + E')\right) - \\ & - 2\,(\varphi'' + 2\,\mathcal{H}\,\varphi')\,\Phi\,\chi - \varphi'\,\chi\,\Phi' - 3\,\varphi'\,\chi\,\psi' \\ & - \frac{1}{4}\,\phi_{ij}\left(\phi_{ij}{}'' + 2\,\mathcal{H}\,\phi_{ij}{}' - \vec{\partial}^2\phi_{ij}\right)\Big] + \\ + \int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\Big[& \frac{1}{a}\chi\,\chi' + \frac{\varphi'}{a}\chi\,\Phi - \partial_\varphi^2 U\,\chi^2 + 2\,M\,\chi\,\vec{D}^2\chi + \\ & + 4\,\partial_\varphi C\left(\chi\,\vec{D}^2\psi + \psi\,\vec{D}^2\chi\right) + \left(3\,\frac{\varphi'}{a} - 6\,\partial_\varphi U\right)\chi\,\psi \end{aligned}$$

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$$+\left(2\,\partial_\varphi U-\frac{\varphi'}{a}\right)\chi\,\vec{\partial}^2E-\frac{6}{a}\,\psi\,\psi\,'-3\,\left(U+2\,\frac{\mathcal{H}}{a}\right)\psi^2-$$
$$-\,6\left(U+2\,\frac{\mathcal{H}}{a}\right)\psi\,\vec{\partial}^2E+4\,C\,\psi\,\vec{D}^2\psi-6\,\frac{\mathcal{H}}{a}\,\psi\,\Phi+$$
$$+\,2\,\frac{\mathcal{H}}{a}\,\Phi\,\vec{\partial}^2E+\frac{2}{a}\,\psi\,\vec{\partial}^2B+\left(U+2\frac{\mathcal{H}}{a}\right)\vec{\partial}^2E\,\vec{\partial}^2E+$$
$$+\,\frac{2}{a}\left(\psi\,'\,\vec{\partial}^2E+\psi\,\vec{\partial}^2E\,'\right)+\frac{1}{4\,a}\,\phi_{ij}\,\phi_{ij}\,'-\frac{1}{2}\left(U+2\,\frac{\mathcal{H}}{a}\right)\phi_{ij}\,\phi_{ij}-$$
$$-\left.\frac{1}{2\,a^2}\,C\,\phi_{ij}\,\vec{\partial}^2\phi_{ij}\right]. \tag{3.30}$$

Terms proportional to the background equations of motion have been cancelled in this expression, but all other terms have been left intact. Specifically, several terms involving U and $\partial_\varphi U$ can be see to cancel after substituting the values $U=-2\mathcal{H}/a$ and $\partial_\varphi U=\varphi\,'/a$. Those terms are left explicit here so that we easily isolate the contributions of the counterterms to the quadratic action. By setting U, M, and C to zero we remove all such contributions, and are left with only those terms that came from the original action (2.1).

The focus of the next section will be explicitly demonstrating that this action is only gauge-invariant when we include the contributions from the counterterms. But let us *momentarily* assume gauge invariance so that we may immediately present the action (3.30) in gauge invariant variables, suitable for application to problems in inflation and cosmology. Since the tensor modes are gauge invariant already we can simply note their contribution to (3.30):

$$S[\phi_{ij}]=-\int_{\mathcal{M}_0}d^4x\sqrt{g}\,\frac{1}{8\,a^2}\,\eta^{\mu\nu}\,\partial_\mu\phi_{ij}\,\partial_\nu\phi_{ij}+\int_{\partial\mathcal{M}_0}d^3x\sqrt{\tilde{g}}\,\frac{C}{4\,a^2}\,\vec{\partial}\phi_{ij}\cdot\vec{\partial}\phi_{ij}\,. \tag{3.31}$$

To obtain this expression we have integrated the tensor terms in (3.30) by parts. Although the tensor modes carry indices, we write their kinetic term using partial derivatives to emphasize the fact that the two polarization states of the tensor modes are equivalent to two minimally coupled, massless scalars.

It is straightforward to write the action for the scalar fluctuations in terms of gauge-invariant variables. Starting with (3.30) the first step is to eliminate two of the scalars Φ,ψ,B,E,χ by making a gauge choice. The variables remaining after gauge fixing are related by two constraints, which can be succinctly expressed in terms of gauge invariant variables as:

$$\begin{aligned}\Psi_B'+\mathcal{H}\,\Phi_B+\frac{1}{2}\,\varphi'\,\chi_B&=0\\ \Phi_B-\Psi_B&=0\,.\end{aligned} \tag{3.32}$$

In an arbitrary gauge these equations take the form:

$$\begin{aligned}\psi'+\mathcal{H}\Phi+\frac{1}{2}\varphi'\chi&=0\\ \Phi-\mathcal{H}(B+E')-(B+E')'&=\psi+\mathcal{H}(B+E')\end{aligned} \tag{3.33}$$

After these constraints are imposed only one physical degree of freedom will remain.

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For example, in comoving gauge, take $\chi_{\rm com} = B_{\rm com} = 0$ and then use the constraints (3.33) to write the remaining expression in terms of the comoving curvature perturbation $\psi_{\rm com}$. The result is:

$$S[\psi_{\rm com}] = -\int_{\mathcal{M}_0} d^4x\sqrt{g}\,\frac{1}{2}\left(\frac{\varphi'}{\mathcal{H}}\right)^2 \nabla^\mu\psi_{\rm com}\,\nabla_\mu\psi_{\rm com} - \\ -\int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\,\frac{1}{a^2}\left(\frac{\varphi'}{\mathcal{H}}\right)^2 \frac{U}{\partial_\varphi U}\,\partial_\varphi C\,\vec{\partial}\psi_{\rm com}\cdot\vec{\partial}\,\psi_{\rm com} \tag{3.34}$$

We can also perform this computation in flat gauge. In this case the action, written in terms of Mukhanov's variable v, becomes:

$$S[v] = \int_{\mathcal{M}_0} d^4x\left[-\frac{1}{2}\,\eta^{\mu\nu}\partial_\mu v\partial_\nu v + \frac{1}{2}\,\frac{z''}{z}\,v^2\right] + \int_{\partial\mathcal{M}_0} d^3x\left[-\frac{1}{2}\,\frac{z'}{z}\,v^2 - \frac{1}{a}\,M\vec{\partial}v\cdot\vec{\partial}v\right] \tag{3.35}$$

where z is defined as $z = a\,\frac{\varphi'}{\mathcal{H}}$. The expressions (3.31), (3.34), and (3.35) for the quadratic action in the presence of a boundary are new results, as far as we are aware.

In our previous paper [7] we computed the quadratic action in longitudinal gauge. The action was then put on-shell by explicitly solving the bulk equations of motion and evaluating the boundary terms on the bulk solution. The equations of motion are difficult to solve in general, which limited our calculation to the case of slow-roll inflation. In the present paper we will instead use symmetries to determine the scaling behavior of the on-shell action, allowing us to partially circumvent the problem of solving the bulk equations explicitly.

3.3 Gauge invariance of the quadratic action

In the preceding subsection we computed the quadratic action in specific gauges, with gauge invariance temporarily assumed. However, as we have emphasized repeatedly, diffeomorphism invariance is not automatic in the presence of a boundary. Indeed, if we neglect counterterms then the quadratic action (3.30) is *not* gauge invariant. The simplest way to demonstrate this is to consider fluctuations which are pure gauge:

$$\begin{aligned} \Phi &= \mathcal{H}\,\delta\tau + (\delta\tau)' \\ \psi &= -\mathcal{H}\,\delta\tau \\ B &= -\varepsilon' + \delta\tau \\ E &= \varepsilon \\ \chi &= -\varphi'\,\delta\tau \end{aligned} \tag{3.36}$$

Inserting these modes in (3.30) gives:

$$\delta^2_\epsilon S = \int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\,\frac{1}{a}\Big[(4\,\mathcal{H}^3 - 8\,\mathcal{H}\,\mathcal{H}' - \mathcal{H}'')\delta\tau^2 - 2\mathcal{H}\,\delta\tau\,\vec{\partial}^2\delta\tau + \\ + 2\,\mathcal{H}\,\vec{\partial}^2\epsilon\,\vec{\partial}^2\epsilon + \left(16\,\mathcal{H}^2 - 4\,\mathcal{H}'\right)\delta\tau\,\vec{\partial}^2\epsilon\Big]\,. \tag{3.37}$$

This expression clearly does not vanish in anything but exceptional circumstances. The failure of the quadratic action to vanish for pure gauge modes exhibits a violation of diffeomorphism invariance.

The introduction of counterterms can rectify this problem. Inserting the gauge modes (3.36) in the quadratic action (3.30) we find that almost all of the terms cancel due to the background equations of motion and the expression (2.32) for the counterterm $U(\varphi)$. The result is now:

$$\delta_\epsilon^2 S_{\rm tot} = -\int_{\partial\mathcal{M}_0} d^3x\sqrt{\tilde{g}}\, 2\,\frac{\mathcal{H}}{a}\left(1 + U\,C - 4\,\partial_\varphi U\,\partial_\varphi C + 2M\,\frac{(\partial_\varphi U)^2}{U}\right)\delta\tau\,\vec{\partial}^2\,\delta\tau \qquad (3.38)$$

Thus, the total action $S_{\rm tot}$ is gauge invariant if the counterterms satisfy the equation:

$$1 + U\,C - 4\,\partial_\varphi U\,\partial_\varphi C + 2M\,\frac{(\partial_\varphi U)^2}{U} = 0 \qquad (3.39)$$

This condition must be satisfied by the counterterms in order that the full quadratic action vanishes for pure gauge modes; i.e. it is a necessary condition for diffeomorphism invariance. We will show below that this condition is satisfied by the counterterms (2.9) and (2.10) that we found by cancelling divergences.

In the preceding subsection we presented the quadratic action in two forms: (3.34) was written in terms of the comoving curvature perturbation $\psi_{\rm com}$, and (3.35) was written in terms of Mukhanov's variable v. These actions should, of course, be equivalent. We can try to verify this by noting that $\psi_{\rm com}$ and v are related as:

$$v = -a\,\frac{\varphi'}{\mathcal{H}}\,\psi_{\rm com}\,. \qquad (3.40)$$

But this simple substitution is *not* enough to relate the actions. Indeed, the action (3.34) depends on the counterterms $U(\varphi)$ and $C(\varphi)\mathcal{R}$, but not $M(\varphi)\vec{D}\varphi\cdot\vec{D}\varphi$. This is because in comoving gauge the spatial inhomogeneties are contained entirely in the fluctuations of the metric, so that the counterterm involving $M(\varphi)$ does not contribute to the action. The action (3.35), on the other hand, depends on the counterterms $U(\varphi)$ and $M(\varphi)\vec{D}\varphi\cdot\vec{D}\varphi$, but not $C(\varphi)\mathcal{R}$. In flat gauge the spatial inhomogeneities are parameterized by $\chi_{\rm flat}$ and the action does not depend on the $C(\varphi)$ counterterm. Therefore, the two actions can agree only if $C(\varphi)$ and $M(\varphi)$ are related. More precisely, if we have:

$$M(\varphi) = \frac{U(\varphi)}{\partial_\varphi U(\varphi)}\,\partial_\varphi C(\varphi) \qquad (3.41)$$

then the two actions are in fact identical upon the substitution (3.40).

The relations (3.39) and (3.41) are conditions that the counterterms must satisfy in order for the total action to be diffeomorphism invariant. We can simplify the condition (3.39) using (3.41) and find:

$$1 + U\,C - 2\,\partial_\varphi U\,\partial_\varphi C = 0 \qquad (3.42)$$

which is precisely (2.9). Furthermore, we can rearrange:

$$\left(1 - \frac{U}{\partial_\varphi U}\,\partial_\varphi\right)(1 + U\,C - 2\,\partial_\varphi U\,\partial_\varphi C) = 0 \qquad (3.43)$$

using (3.41) and then recover (2.10). *Diffeomorphism invariance of the total action guarantees that the counterterms satisfy the equations* (2.8)*,* (2.9)*,* (2.10)*, which were originally motivated by the cancellation of divergences.*

One might wonder if these conditions on the counterms, necessary for diffeomorphism invariance, are in fact *sufficient* to guarantee a diffeomorphism invariant action. The most straightforward way to show that this is indeed the case is to use the expressions (3.39) and (3.41) in the action (3.30), where the gauge has not been fixed. Though tedious, after simplifying the action using the constraints (3.32) the various terms combine to form gauge invariant variables. As a check of our calculations, the actions in comoving gauge (3.34) and flat gauge (3.35) were both obtained in this manner.

Let us summarize the main results of our discussions of counterterms at different points in sections 2 and 3. There are (at least) two ways to motivate the addition of local counterterms on the boundary, and determine their form:

1. The on-shell action diverges as $\tau_0 \to 0$. The counterterms are needed to cancel these divergences. The H-J equation shows that the functions $U(\varphi), M(\varphi), C(\varphi)$ defining the counterterms must satisfy eqs. (2.8), (2.9), (2.10).

2. Diffeomorphism invariance cannot be taken for granted in the presence of a boundary. Counterterms are needed to restore diffeomorphism invariance. We implement diffeomorphism invariance by imposing cancellation of boundary sources, vanishing action for fluctuations of pure gauge form, and equivalence of boundary actions in comoving and flat gauges; these imply that the counterterms must satisfy eqs. (2.32), (3.39), and (3.41).

The three conditions imposed on counterterms under (1) are equivalent to those imposed under (2). We can thus take either (1) or (2) as the primary motivation, the other will then follow. More precisely, starting with either (1) or (2), the other is in fact a consistency condition: the H-J equation is a manifestation of the hamiltonian constraint, so diffeomorphism invariance is assumed, albeit not emphasized, in (1). Similarly, once we have imposed diffeomorphism invariance in (2), the hamiltonian constraint is satisfied and so divergences are cancelled. The real lesson is that the counterterms are an essential ingredient for a consistent gravitational theory.

4. The holographic renormalization group

In this section we exploit diffeomorphism invariance to constrain the functional form of the on-shell action and its functional derivatives. An especially in important class of four dimensional diffeomorphisms act as Weyl rescalings of the boundary data. These ensure that the on-shell action satisfies a renormalization-group equation and that second order terms in the action, describing fluctuations around the background, obey a Callan-Symanzik equation. These constraints on the action suggest an analogy between inflation and a three-dimensional euclidean field theory near a renormalization group fixed point, with the on-shell action interpreted as the generating functional for correlators of field theory operators sourced by the boundary data.

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4.1 The master equation

The on-shell, renormalized action is a functional of the induced metric[6] $\tilde{g}_{ij}(\vec{x},\tau)$ and the field $\varphi(\vec{x},\tau)$ evaluated on the spacelike hypersurface $\tau = \tau_0$:

$$S_{\text{tot}} = S_{\text{tot}} \left[\varphi(\vec{x},\tau_0), \tilde{g}_{ij}(\vec{x},\tau_0)\right] \tag{4.1}$$

In the following sections we will use the notation $\varphi(\vec{x})$ to refer to $\varphi(\vec{x},\tau_0)$, and similarly for $\tilde{g}_{ij}(\vec{x})$.

Diffeomorphism invariance requires that the transformation of S_{tot} under an infinitesimal four dimensional diffeomorphism vanishes:

$$\delta_\epsilon S_{\text{tot}} = \int d^3x \left[\delta_\epsilon \tilde{g}_{ij} \frac{\delta S_{\text{tot}}}{\delta \tilde{g}_{ij}} + \delta_\epsilon \varphi \frac{\delta S_{\text{tot}}}{\delta \varphi}\right] = 0\,. \tag{4.2}$$

The transformation laws of the fields (2.19) and (2.20) can be written in terms of transverse and normal components as:

$$\begin{aligned} \delta_\epsilon \varphi &= n_\lambda \epsilon^\lambda\, n^\mu \nabla_\mu \varphi - \epsilon^i D_i \varphi \\ \delta_\epsilon \tilde{g}_{ij} &= -D_i \epsilon_j - D_j \epsilon_i - 2\, n_\lambda \epsilon^\lambda K_{ij} \end{aligned} \tag{4.3}$$

The first two terms in the transformation of $\tilde{g}_{ij}$ are due to the spacelike part of the diffeomorphism ϵ^i. The last term appears because the extrinsic curvature is essentially the normal derivative of the induced metric on a hypersurface. The transformation of the on-shell action becomes:

$$\begin{aligned} \delta_\epsilon S_{\text{tot}} = \int d^3x \Bigg[&- 2 D_i \epsilon_j(\vec{x}) \frac{\delta S_{\text{tot}}}{\delta \tilde{g}_{ij}(\vec{x})} - \epsilon^i(\vec{x})\, D_i \varphi(\vec{x}) \frac{\delta S_{\text{tot}}}{\delta \varphi(\vec{x})} + \\ &+ n_\lambda \epsilon^\lambda(\vec{x}) \left(n^\mu \nabla_\mu \varphi(\vec{x}) \frac{\delta S_{\text{tot}}}{\delta \varphi(\vec{x})} - 2\, K_{ij} \frac{\delta S_{\text{tot}}}{\delta \tilde{g}_{ij}(\vec{x})} \right) \Bigg] = 0\,. \end{aligned} \tag{4.4}$$

The field transformations can be written as:

$$\delta_\epsilon \varphi(\vec{x}) = -\epsilon^i\, D_i \varphi(\vec{x}) - \lambda_\epsilon\, \beta(\varphi(\vec{x})) \tag{4.5}$$

$$\delta_\epsilon \tilde{g}_{ij} = -D_i \epsilon_j - D_j \epsilon_i - 2\, \lambda_\epsilon\, \tilde{g}_{ij} \tag{4.6}$$

where $\lambda_\epsilon = \mathcal{H}\, \delta\tau$ parametrizes the variation of the normal, and we defined $\beta(\varphi)$ as:

$$\beta(\varphi) = \frac{1}{\mathcal{H}} \frac{\partial \varphi}{\partial \tau}\,. \tag{4.7}$$

Expressed this way, the transformation properties of $\varphi(\vec{x})$ and $\tilde{g}_{ij}(\vec{x})$ under a four dimensional diffeomorphism have a clear interpretation on the $\tau = \tau_0$ hypersurface: diffeomorphisms involving the direction normal to the hypersurface are realized as Weyl rescalings,

[6] In the remainder of the paper we will allow a slightly different notation for the induced metric and other tensors intrinsic to the $\tau = \tau_0$ hypersurface. In order to highlight the difference between the directions normal to and parallel to the boundary, we will use latin indices $i, j, \ldots$ on intrinsic tensors. This should *not* be interpreted as four dimensional tensors simply restricted to their spatial components.

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while diffeomorphisms involving directions along the hypersurface are interpreted as three-dimensional diffeomorphisms. The transformation of the on-shell action under diffeomorphisms can now be written:

$$\delta_\epsilon S_{\rm tot} = \int d^3x \bigg[- 2D_i\epsilon_j(\vec{x})\, \frac{\delta S_{\rm tot}}{\delta \tilde{g}_{ij}(\vec{x})} - \epsilon^i(\vec{x})\, D_i\varphi(\vec{x})\, \frac{\delta S_{\rm tot}}{\delta\varphi(\vec{x})} - \\ - \lambda_\epsilon(\vec{x}) \left(\beta(\varphi(\vec{x}))\, \frac{\delta S_{\rm tot}}{\delta\varphi(\vec{x})} + 2\,\tilde{g}_{ij}(\vec{x})\, \frac{\delta S_{\rm tot}}{\delta\tilde{g}_{ij}(\vec{x})} \right) \bigg] = 0 \tag{4.8}$$

This is our master equation which constrains the dependence of $S_{\rm tot}$ on the fields $\varphi(\vec{x})$ and $\tilde{g}_{ij}(\vec{x})$. We simplified the equation by writing the extrinsic curvature as $K_{ij} = (\mathcal{H}/a)\,\tilde{g}_{ij}$.

As it stands the master equation (4.8) is a functional differential equation for the action $S_{\rm tot}$. Its meaning becomes more transparent when we exploit the gauge invariance of $S_{\rm tot}$ and choose a simple gauge. For example, in the flat gauge described in section 3.1.2, the spatial inhomogeneities due to fluctuations around the background are contained entirely in the scalar field:[7]

$$\varphi_f(\vec{x}) = \varphi + \chi_f(\vec{x})\,. \tag{4.9}$$

In this gauge the induced metric on $\partial\mathcal{M}_0$ is simply $\tilde{g}_{ij} = a(\tau_0)^2\delta_{ij}$; so, in flat gauge, the total action is a functional of the scalar $\varphi_f(\vec{x})$ and a *function* of the scale factor $a(\tau_0)$. We can expand the action as a series in $\chi_f(\vec{x})$:

$$S_{\rm tot}[a,\varphi_f(\vec{x})] = S^{(0)}_{\rm tot} + \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} d^3x_i\, \frac{1}{n!}\, S^{(n)}_{\rm tot}[a,\varphi;\vec{x}_1,\ldots,\vec{x}_n]\, \chi_f(\vec{x}_1)\ldots\chi_f(\vec{x}_n) \tag{4.10}$$

where the coefficient $S^{(n)}_{\rm tot}$ is the n^{th} functional derivative of $S_{\rm tot}$ with respect to $\varphi_f(\vec{x})$, evaluated at $\phi_f(\vec{x}) = \phi$ (or $\chi_f(\vec{x}) = 0$):

$$S^{(n)}_{\rm tot}[a,\varphi;\vec{x}_1,\ldots,\vec{x}_n] = \left.\frac{\delta^n\, S_{\rm tot}}{\delta\varphi_f(\vec{x}_1)\ldots\delta\varphi_f(\vec{x}_n)}\right|_{\chi_f=0}. \tag{4.11}$$

The first term in the expansion, $S^{(0)}_{\rm tot}$, is simply the action for the background; i.e. the part of the action with no dependence on the fluctuations. Furthermore, since the action is on-shell its first order variation must vanish for arbitrary fluctuations around the background, implying $S^{(1)}_{\rm tot} = 0$.

In the following we will write the master equation (4.8) in more explicit forms for specific choices of the diffeomoprphism ϵ^μ. This will lead to differential equations that constrain the dependence of the coefficients $S^{(n)}_{\rm tot}$ on the homogeneous background fields a, φ, and the n spatial points $\vec{x}_i$.

[7]Here we have again used a notation that leaves the dependence on τ_0 implicit, with φ representing the background value of the scalar.

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4.1.1 Weyl rescalings

First we consider a diffeomorphism that only involves the normal direction and is independent of the spatial coordinates. Then $\vec{\epsilon} = 0$ and λ_ϵ depends only on τ_0. Under this diffeomorphism the master equation (4.8) becomes:

$$\delta_\epsilon S_{\rm tot} = -\lambda_\epsilon \int d^3z \left(2\, \tilde{g}_{ij}(\vec{z})\, \frac{\delta S_{\rm tot}}{\delta \tilde{g}_{ij}(\vec{z})} + \beta(\varphi(\vec{z}))\, \frac{\delta S_{\rm tot}}{\delta \varphi(\vec{z})} \right) \;=\; 0 \tag{4.12}$$

In flat gauge the action is a *function* of the scale factor $a(\tau_0)$ and a functional of $\varphi_f(\vec{x})$; so it is more appropriate to write (4.12) as:

$$\left(a\, \frac{\partial}{\partial a} + \int d^3z\, \beta(\varphi_f(\vec{z}))\, \frac{\delta}{\delta \varphi_f(\vec{z})} \right) S_{\rm tot} = 0 \tag{4.13}$$

We evaluate this equation for the background by setting the fluctuations equal to 0. In this case the complete scalar field φ_f reduces to the homogeneous background φ and we can treat $S_{\rm tot}$ as a function of φ, rather than a functional. We can therefore write (4.13) as:

$$\left(a\, \frac{\partial}{\partial\, a} + \beta(\varphi)\, \frac{\partial}{\partial \varphi} \right) S^{(0)}_{\rm tot} = 0\,. \tag{4.14}$$

Equation (4.14) can be interpreted as an RG equation for $S^{(0)}_{\rm tot}$. The field φ plays the role of a coupling with beta function $\beta(\varphi)$, and $a(\tau_0)$ is the scale factor. The β-function was defined in (4.7) as $\beta = \mathcal{H}^{-1}\partial_\tau \varphi$ so the RG equation (4.14) states that the τ_0-dependence of $S^{(0)}_{\rm tot}$ through the scale factor $a(\tau_0)$ is balanced by the dependence of τ_0 through the background scalar field $\varphi(\tau_0)$. We can express this scale independence as:

$$\frac{dS^{(0)}_{\rm tot}}{da} = 0 \tag{4.15}$$

or, alternatively, as independence of the cut-off τ_0:

$$\frac{dS^{(0)}_{\rm tot}}{d\tau_0} = 0\,. \tag{4.16}$$

We want to find additional differential equations which constrain the higher coefficients $S^{(n)}_{\rm tot}$ in the expansion (4.10) of the action. To do so we must vary (4.12) with respect to the field φ_f, before taking $\chi_f = 0$. Taking one functional derivative of (4.12) with respect to $\varphi_f(\vec{x})$ yields the functional equation:

$$W_1[\varphi_f(\vec{x})] \cdot \frac{\delta S_{\rm tot}}{\delta \varphi_f(\vec{x})} = 0 \tag{4.17}$$

where $W_1[\varphi_f(\vec{x})]$ has the form:

$$W_1[\varphi_f(\vec{x})] = a\, \frac{\partial}{\partial\, a} + \int d^3z \left(\beta(\varphi_f(\vec{z}))\, \frac{\delta}{\delta \varphi_f(\vec{z})} + \delta(\vec{z} - \vec{x})\, \frac{\delta \beta(\varphi_f(\vec{z}))}{\delta \varphi_f(\vec{z})} \right). \tag{4.18}$$

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Evaluating (4.17) for the background by setting $\chi_f(\vec{x}) = 0$, we can replace the functional derivatives by ordinary derivatives, and find the differential equation:

$$\left(a \frac{\partial}{\partial a} + \beta(\varphi) \frac{\partial}{\partial \varphi} + \gamma(\varphi) \right) S^{(1)}_{\mathrm{tot}}[a, \varphi; \vec{x}] = 0 \tag{4.19}$$

where we defined the anomalous dimension as:

$$\gamma(\varphi) \equiv \frac{\partial \beta(\varphi)}{\partial \varphi} \,. \tag{4.20}$$

Equation (4.19) takes the form of a Callan-Symanzik equation in which the function $S^{(1)}_{\mathrm{tot}}$ plays the role of a one-point function of an operator with anomalous dimension γ. Of course this equation is actually trivial since, as discussed in the previous subsection, the expansion for S_{tot} does not contain a linear term and so $S^{(1)}_{\mathrm{tot}} = 0$.

However, we can repeat the procedure by taking one more functional derivative of (4.17) with respect to $\varphi_f(\vec{y})$ and evaluating the resulting functional differential equation at $\chi_f = 0$. This yields an equation satisfied by $S^{(2)}_{\mathrm{tot}}$:

$$\left(a \frac{\partial}{\partial a} + \beta(\varphi) \frac{\partial}{\partial \varphi} + 2\gamma(\varphi) \right) S^{(2)}_{\mathrm{tot}}[a, \varphi; \vec{x}, \vec{y}] = 0 \,. \tag{4.21}$$

A contact term proportional to $\delta(\vec{x} - \vec{y}) \partial^2_\varphi \beta$ was omitted because it acts on $S^{(1)}_{\mathrm{tot}}$ and so vanishes. Just as equation (4.19) can be thought of as the Callan-Symanzik equation for the one-point function of an operator with anomalous dimension γ, equation (4.21) can be regarded as the Callan-Symanzik equation for the two-point function of the same operator. In this case, $S^{(2)}_{\mathrm{tot}}$ corresponds to the two-point function.

4.1.2 Three dimensional diffeomorphisms

A particularly simple case of the master equation (4.8) corresponds to three dimensional diffeomorphisms. These are generated by four dimensional diffeomorphisms with:

$$n_\mu \epsilon^\mu = 0 \,. \tag{4.22}$$

In this case (4.8) becomes:

$$\delta_{\vec{\epsilon}} S_{\mathrm{tot}} = \int d^3x \left[-2 D_i \epsilon_j \frac{\delta S_{\mathrm{tot}}}{\delta \tilde{g}_{ij}} - \epsilon^i \, D_i \varphi \, \frac{\delta S_{\mathrm{tot}}}{\delta \varphi} \right] = 0 \,. \tag{4.23}$$

Integrating the first term by parts and requiring the integrand to vanish for arbitrary $\vec{\epsilon}$ leads to:

$$2 \, D_j \left(\frac{\delta S_{\mathrm{tot}}}{\delta \tilde{g}_{ij}} \right) - D^i \varphi \, \frac{\delta S_{\mathrm{tot}}}{\delta \varphi} = 0 \,. \tag{4.24}$$

In the Hamilton-Jacobi formalism functional derivatives of the on-shell action with respect to the boundary data yield the canonical momenta, evaluated at the boundary. It is straightforward to see that (4.24) is simply the usual momentum constraint that appears in the canonical $3+1$ treatment of four dimensional gravity. Here it represents invariance of the total action under reparameterizations of the spatial coordinates.

4.1.3 Conformal transformations

Weyl rescalings are diffeomorphisms that act only on the normal coordinate; and three dimensional diffeomorphisms are bulk diffeomorphisms acting only on the spatial coordinates. We can also consider four dimensional diffeomorphisms for which the induced three dimensional diffeomorphism compensates for a Weyl rescaling of the metric $\tilde{g}_{ij}$ in such a way that $\delta_\epsilon \tilde{g}_{ij} = 0$. These diffeomorphisms act as conformal transformations on the boundary.

Under a general diffeomorphism the induced metric on $\partial\mathcal{M}_0$ transforms as:

$$\delta_\epsilon \tilde{g}_{ij} = -D_i\epsilon_j - D_j\epsilon_i - 2\,\lambda_\epsilon \tilde{g}_{ij} = 0 \tag{4.25}$$

Therefore, the condition for $\delta_\epsilon \tilde{g}_{ij} = 0$ is:

$$\begin{aligned} D_i\epsilon_j + D_j\epsilon_i \;&=\; -2\,\lambda_\epsilon\,\tilde{g}_{ij} \\ D_i\epsilon^i \;&=\; -3\,\lambda_\epsilon \end{aligned} \tag{4.26}$$

For this type of diffeomorphisms the master equation (4.8) becomes:

$$\delta_\epsilon S_{\rm tot} = -T_\epsilon\,[\varphi]\cdot S_{\rm tot} = 0 \tag{4.27}$$

where $T_\epsilon\,[\varphi]$ is the functional differential operator:

$$T_\epsilon\,[\varphi] = \int d^3x\,\left[\epsilon^i(\vec{x})D_i\varphi + \lambda_\epsilon(\vec{x})\,\beta(\varphi)\right]\frac{\delta}{\delta\varphi} \tag{4.28}$$

Applying *two* functional derivatives, with respect to $\varphi(\vec{x})$ and $\varphi(\vec{y})$, we find:

$$T_\epsilon\,[\vec{x},\vec{y},\varphi]\cdot\frac{\delta^2 S_{\rm tot}}{\delta\varphi(\vec{x})\,\delta\varphi(\vec{y})} = 0 \tag{4.29}$$

where:

$$T_\epsilon\,[\vec{x},\vec{y},\varphi] = T_\epsilon[\varphi] - \epsilon^i(\vec{x})\frac{\partial}{\partial x^i} - \epsilon^i(\vec{y})\,\frac{\partial}{\partial y^i} + \lambda_\epsilon(\vec{x})\left(3 + \frac{\delta\beta(\varphi(\vec{x}))}{\delta\varphi(\vec{x})}\right) + \lambda_\epsilon(\vec{y})\left(3 + \frac{\delta\beta(\varphi(\vec{y}))}{\delta\varphi(\vec{y})}\right). \tag{4.30}$$

This operator, acting the second functional derivative of $S_{\rm tot}$, satisfies:

$$\Big[T_{\epsilon_1}\,[\vec{x},\vec{y},\varphi]\,,T_{\epsilon_2}\,[\vec{x},\vec{y},\varphi]\Big] = T_{[\epsilon_1,\epsilon_2]}\,[\vec{x},\vec{y},\varphi] \tag{4.31}$$

In other words it satisfies the Lie algebra associated with conformal transformations.

As a simple application of the conformal group, consider (4.29) for constant λ_ϵ which implies $\epsilon^i = -\lambda_\epsilon\,x^i$. Taking $\chi_f(\vec{x}) = 0$ we find:

$$\left(s^i\,\frac{\partial}{\partial s^i} + 6 + 2\gamma + \beta(\varphi)\,\frac{\partial}{\partial\varphi}\right)\,S^{(2)}_{\rm tot}[a,\varphi;\vec{s}] = 0 \tag{4.32}$$

where $s^i = x^i - y^i$. This is the position-space version of the Callan-Symanzik equation (4.21). A similar computation demonstrates that $S^{(2)}_{\rm tot}[a,\varphi;\vec{s}]$ is invariant under rotations.

4.2 Inflation as a holographic quantum field theory

The terminology we have introduced in this section to describe the constraints due to diffeomorphism invariance is designed to suggest a formal analogy between the renormalized on-shell action for inflation and a three-dimensional euclidean quantum field theory near a UV fixed point. The analogy is implemented by the map:

$$\Psi[\varphi(\vec{x}), \tilde{g}_{ij}(\vec{x})] = Z[\varphi(\vec{x}), \tilde{g}_{ij}(\vec{x})] \tag{4.33}$$

where the semi-classical wave function is constructed from the renormalized on-shell action $S_{\rm tot}$, with boundary data $\varphi(\vec{x})$ and $\tilde{g}_{ij}(\vec{x})$:

$$\Psi[\varphi(\vec{x}), \tilde{g}_{ij}(\vec{x})] = \exp\left(i\, S_{\rm tot}[\varphi(\vec{x}), \tilde{g}_{ij}(\vec{x})]\right) \tag{4.34}$$

and the partition function refers to a euclidean QFT with sources χ and h_{ij}:

$$Z[\varphi(\vec{x}), \tilde{g}_{ij}(\vec{x})] = \left\langle \exp\left(\int d^3x \left[\frac{1}{2} h_{ij}(\vec{x})\, T^{ij}(\vec{x}) + \chi(\vec{x})\, \mathcal{O}(\vec{x})\right]\right)\right\rangle. \tag{4.35}$$

The brackets in equation (4.35) denote the QFT expectation value, with the background fields representing the couplings in the unperturbed theory. The semi-classical wave functional for inflation is then interpreted as the generating functional for correlators of the operator $\mathcal{O}$ and the associated stress-energy tensor T^{ij} in a quantum field theory. This hypothetical theory has been displaced from the fixed point by an operator $\mathcal{O}$ that is sourced by the boundary data $\varphi(\vec{x}, \tau_0)$.

According to the identification (4.33) we can calculate correlators of the QFT operators by functionally differentiating the partition function with respect to the sources, and then setting the sources equal to zero:

$$\langle \mathcal{O}(\vec{x}_1) \dots \mathcal{O}(\vec{x}_n)\rangle \equiv (-i)^n \left. \frac{\delta \Psi}{\delta\varphi_f(\vec{x}_1) \dots \delta\varphi_f(\vec{x}_n)}\right|_{\chi=0} = S^{(n)}_{\rm tot}[a, \varphi; \vec{x}_1, \dots, \vec{x}_n] \tag{4.36}$$

where the last equality utilized the definition (4.10) of the coefficient $S^{(n)}_{\rm tot}$. An immediate consequence of this relation is that the one-point function $\langle \mathcal{O}(\vec{x})\rangle$ vanishes, because there is no linear term in the expansion of the on-shell action. The differential equations for $S^{(n)}_{\rm tot}$ generated by bulk diffeomorphisms become, according to the analogy, equations that are familiar from renormalization theory. For example, the RG equation (4.14) for the background part of the on-shell action becomes an RG equation for the field theory partition function with no sources:

$$\left(a\, \frac{\partial}{\partial a} + \beta(\varphi)\, \frac{\partial}{\partial \varphi}\right) Z[a, \varphi] = 0 \tag{4.37}$$

and the equation (4.21) that governs the coefficient of the quadratic term in the on-shell action becomes a Callan-Symanzik equation for the two-point function of the operator $\mathcal{O}(\vec{x})$:

$$\left(a\, \frac{\partial}{\partial a} + \beta(\varphi)\, \frac{\partial}{\partial \varphi} + 2\gamma(\varphi)\right) \langle \mathcal{O}(\vec{x})\, \mathcal{O}(\vec{y})\rangle = 0\,. \tag{4.38}$$

Strominger has proposed a relation similar to (4.33) as a full-fledged quantum duality [21, 22], known as the dS/CFT correspondence. This duality, if true, would relate a gravitational theory on (asymptotically) de Sitter spacetime to a (deformed) quantum conformal field theory on the boundary. Our approach is different both in its technical details (which follow Maldacena [6]) and in ambition. We regard (4.33) as a convenient analogy which allow us to exploit much of our intuition about RG flows, because the physics of the inflating spacetime is governed by the same set of RG and Callan-Symanzik equations. However, similarities between the two systems does not, in and of itself, justify the idea of a true duality, and we do not appeal to a notion of a microscopically defined QFT dual to inflation.

4.3 The Ward identity

Diffeomorphisms in the normal direction, which result in Weyl rescalings of the boundary data, constrain $S_{\rm tot}$ according to (4.12):

$$\int d^3x \left[2\,\tilde{g}_{ij}\,\frac{\delta S_{\rm tot}}{\delta \tilde{g}_{ij}} + \beta(\varphi)\,\frac{\delta S_{\rm tot}}{\delta\varphi}\right] = 0 \tag{4.39}$$

The map (4.33) then gives a relation between the operator $\mathcal{O}$ and the trace of the stress-energy tensor T^i_i:

$$T^i_i + \beta\,\mathcal{O} = 0 \tag{4.40}$$

This is the conformal Ward identity for the field theory. For a conformally invariant theory the trace of the stress energy tensor vanishes.[8] The operator $\mathcal{O}$ displaces the field theory from its scale-invariant fixed point, and T^i_i is no longer vanishing. Instead, it is proportional to the β-function of the operator generating the flow away from the fixed point.

The Conformal Ward Identity ensures that the terms involving scalars form gauge invariant combinations:

$$\frac{1}{2}\,h_{ij}T^{ij} + \chi\,\mathcal{O} = \psi\,T^i_i + \chi\,\mathcal{O} = (\chi - \beta\psi)\,\mathcal{O} \;=\; \chi_f(\vec{x})\,\mathcal{O}(\vec{x}) \tag{4.41}$$

$$= -\psi_{\rm com}(\vec{x})\,T^i_i(\vec{x}) \tag{4.42}$$

This is because the β-function appearing in (4.40) is the same factor that relates the gauge invariant variables $\psi_{\rm com}$ and $\chi_{\rm flat}$:

$$\chi_{\rm flat}(\vec{x}) = -\frac{\varphi'}{\mathcal{H}}\,\psi_{\rm com}(\vec{x}) = -\beta\,\psi_{\rm com}(\vec{x}) \tag{4.43}$$

Thus, the scalar fluctuation in flat gauge, $\chi_f(\vec{x})$, sources the operator $\mathcal{O}(\vec{x})$, and the co-moving curvature perturbation $\psi_{\rm com}(\vec{x})$ sources the trace of the stress-energy tensor. We already used this when deriving (4.36).

[8]Inflation in four dimensions is analagous to a euclidean field theory in three dimensions, so there is no conformal anomaly to consider.

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The Ward Identity (4.40) holds as an operator equation, as opposed to a relation that applies to expectation values of the operators. This is because it is a consequence of the full master equation (4.8), rather than the differential equations that constrain the coefficients of the Taylor expanded action. We can therefore use it to relate higher correlators, such as:

$$\langle T^i_i(\vec{x})\, T^j_j(\vec{y})\rangle = \beta(\varphi)^2\, \langle \mathcal{O}(\vec{x})\, \mathcal{O}(\vec{y})\rangle \tag{4.44}$$

This is useful for the derivation of the CMB spectrum, presented in the next section.

In addition to the conformal Ward identity we can obtain a Ward Identity associated with three dimensional diffeomorphisms. It follows from the momentum constraint (4.24):

$$D^j\, T_{ij} + D_i\varphi\, \mathcal{O} = 0 \tag{4.45}$$

Like the conformal Ward Identity this is an operator equation that holds outside of expectation values.

5. Renormalization group improved CMB power spectrum

In this section we integrate the Callan-Symanzik equation and find the explicit form for the two point correlators which satisfy the constraints imposed by diffeomorphism invariance. Our results reproduce the familiar first-order result of slow-roll inflation and organize higher order corrections in analogy with RG-improved perturbation theory in quantum field theory. We consider scalar and tensor modes in turn.

5.1 The scalar modes

After expanding the action in terms of fluctuations around the inflationary background, correlation functions of gauge invariant variables can be computed in terms of functional integrals weighted by the modulus-squared of the wave-functional $\Psi[\varphi, \tilde{g}_{ij}] \sim e^{iS_{\rm tot}[\varphi,\tilde{g}_{ij}]}$. The simplest example is the two-point correlation function of the scalar field in flat gauge. Since the leading nontrivial terms in the action are quadratic:

$$\begin{aligned} S_{\rm tot}[a, \varphi_f(\vec{x})] &= S^{(0)}_{\rm tot} + \frac{1}{2}\int d^3x \int d^3y\, S^{(2)}_{\rm tot}[a, \varphi, |\vec{x}-\vec{y}|\,]\, \chi_f(\vec{x})\, \chi_f(\vec{y}) + \cdots && (5.1)\\ &= S^{(0)}_{\rm tot} + \frac{1}{2}\int d^3k\, \tilde{S}^{(2)}_{\rm tot}[a, \varphi, k\,]\ \tilde{\chi}_f(\vec{k})\, \tilde{\chi}_f(-\vec{k}) + \cdots && (5.2)\end{aligned}$$

where Fourier modes are introduced as:

$$\tilde{\chi}_f(\vec{k}) = \int d^3x\, \frac{e^{i\vec{k}\cdot\vec{x}}}{(2\pi)^{3/2}}\, \chi_f(\vec{x}) \tag{5.3}$$

the functional integral is, to the leading order, a simple gaussian. It gives [6]:

$$\langle \tilde{\chi}_f(\vec{k})\, \tilde{\chi}_f(-\vec{k})\rangle = \int \mathcal{D}\tilde{\chi}_f(\vec{q})\tilde{\chi}_f(\vec{k})\tilde{\chi}_f(-\vec{k})\ |\Psi[\chi_f(\vec{q})]|^2 = \frac{1}{2\,{\rm Im}\,\tilde{S}^{(2)}_{\rm tot}[a,\varphi,k\,]}\,. \tag{5.4}$$

The lorentzian path integral is interpreted, as usual, by continuation from euclidean space. The imaginary part of the quadratic term in the action, ${\rm Im}\, S^{(2)}_{\rm tot}$, is therefore the (negative of the) real part of the euclidean action. Using the analogy between inflation and a QFT near a renormalization group fixed point outlined in the previous section, we will denote $\mathrm{Im}\, S^{(2)}_{\rm tot}$ by the two-point function $-\mathrm{Re}\langle\mathcal{O}(\vec{k})\mathcal{O}(-\vec{k})\rangle$. The correlator of the scalar fluctuation can therefore be written as:

$$\langle\chi_f(\vec{k})\,\chi_f(-\vec{k})\rangle = -\frac{1}{2\mathrm{Re}\langle\mathcal{O}(\vec{k})\mathcal{O}(-\vec{k})\rangle}\,. \tag{5.5}$$

Note that the counterterms are real terms in the lorentzian action, and so they do not contribute to the modulus of the wave function [6].

As discussed in section 4, diffeomorphism invariance implies that the two-point correlation function of the operator $\mathcal{O}$ is constrained by the Callan-Symanzik equation (4.38):

$$\left(a\frac{\partial}{\partial a} + \beta(\varphi)\frac{\partial}{\partial\,\varphi} + 2\gamma(\varphi)\right)\,\langle\mathcal{O}(\vec{x})\,\mathcal{O}(\vec{y})\rangle = 0 \tag{5.6}$$

This equation can be solved by a procedure that is standard in renormalization group theory [20]. We first use dimensional analysis on the three-dimensional boundary to write the correlator in the form:

$$\langle\mathcal{O}(\vec{x})\,\mathcal{O}(\vec{y})\rangle = \frac{1}{|\vec{x}-\vec{y}|^6}F(|\vec{x}-\vec{y}|a,\varphi) \tag{5.7}$$

To arrive at this result we also appeal to rotational invariance and recall that $|\vec{x}-\vec{y}|a$ is the physical length, as opposed to the coordinate length $|\vec{x}-\vec{y}|$. It is useful to Fourier transform the position space Callan-Symanzik equation (4.32):

$$\left(3 - \vec{k}\cdot\frac{\partial}{\partial\vec{k}} + \beta(\varphi)\frac{\partial}{\partial\,\varphi} + 2\gamma(\varphi)\right)\langle\mathcal{O}(\vec{k})\,\mathcal{O}(-\vec{k})\rangle = 0 \tag{5.8}$$

and the *ansatz* (5.7):

$$\langle\mathcal{O}(\vec{k})\,\mathcal{O}(-\vec{k})\rangle = k^3\,\tilde{F}\left(\frac{k}{a},\varphi\right) \tag{5.9}$$

The ratio k/a is the physical momentum. The next step in solving the Callan-Symanzik equation is to introduce a "running coupling" $\bar{\varphi}(k/a,\varphi)$ defined by:

$$k\,\frac{\partial\bar{\varphi}}{\partial\,k} = \beta(\bar{\varphi}) \tag{5.10}$$

In the present context the "running coupling" is just the rolling scalar field of the background, evolving according to the FRW equations. The running coupling follows φ along the renormalization group flow to an *arbitrary* reference scale M, with $\bar{\varphi}(M,\varphi) = \varphi$.[9] Employing (5.9) and (5.10) in solving (5.8) we find that $\langle\mathcal{O}(\vec{k})\,\mathcal{O}(-\vec{k})\rangle$ must take the form:

$$\langle\mathcal{O}(\vec{k})\,\mathcal{O}(-\vec{k})\rangle = k^3\,\tilde{F}_0\left(\bar{\varphi}\left(k/a,\varphi\right)\right)\exp\left[\int_{aM}^{k} d\log\left(\frac{k'}{aM}\right)2\,\gamma\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)\right] \tag{5.11}$$

[9] More explicitly, if the value of the scalar field is φ when the physical momentum is k/a, then $\bar{\varphi}(k/a,\varphi)$ was the value of the field when the physical momentum was equal to the reference scale, $k/a = M$.

where $\tilde{F}_0$ is some unknown function of the running coupling $\bar{\varphi}$. In verifying this it is useful to note that the running coupling satisfies:

$$\beta(\varphi)\frac{\partial\bar{\varphi}}{\partial\varphi} = \beta(\bar{\varphi})\,. \tag{5.12}$$

To summarize the computation thus far, we have used the Callan-Symanzik equation to write the two-point correlator of the scalar field as:

$$\langle\tilde{\chi}_f(\vec{k})\,\tilde{\chi}_f(-\vec{k})\rangle = \frac{1}{2k^3\tilde{F}_0\left(\bar{\varphi}\left(k/a,\varphi\right)\right)}\exp\left[-\int_{aM}^{k} d\log\left(\frac{k'}{aM}\right)2\gamma\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)\right]. \tag{5.13}$$

The scalar field $\chi_f(x)$ has mass dimension 1 in four dimensions; and so the function $\tilde{F}_0$ has mass dimension -2. In order to extract this remaining dimensionful factor it is useful to introduce a running Hubble constant $H(\bar{\varphi})$ which takes the form:

$$H\left(\bar{\varphi}\left(\frac{k}{a},\varphi\right)\right) = H(\varphi)\exp\left(-\frac{1}{2}\int_{aM}^{k} d\log\left(\frac{k'}{aM}\right)\beta\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)^2\right) \tag{5.14}$$

This expression can be verified by differentiating with respect to φ on both sides and using the relations (5.10) and (5.12) to simplify the derivatives, as well as the relation:

$$\beta(\varphi) = -2\,\frac{\partial_\varphi H}{H} \tag{5.15}$$

which is easily derived from the FRW equations and the relation $H = \mathcal{H}/a$ between the physical Hubble factor H and the conformal Hubble factor $\mathcal{H}$. We can now extract the dimensionful factor from the correlator (5.13) and write it as:

$$\begin{aligned}\langle\tilde{\chi}_{\text{flat}}(\vec{k})\tilde{\chi}_{\text{flat}}(-\vec{k})\rangle \;=\; &\frac{H(\varphi)^2}{2k^3}\mathcal{A}_s\left(\bar{\varphi}\left(\frac{k}{a},\varphi\right)\right)\exp\left(-\int_{aM}^{k} d\log\left(\frac{k'}{aM}\right)\times\right.\\ &\left.\times\left(\beta\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)^2+2\gamma\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)\right)\right)\end{aligned} \tag{5.16}$$

where $\mathcal{A}_s$ is a dimensionless amplitude that represents dynamical input. It is not determined by the scaling arguments used in the Callan-Symanzik analysis.

In cosmology it is natural to compute correlation functions of the comoving curvature perturbation,[10] $\psi_{\text{com}} = -\beta(\varphi)^{-1}\chi_{\text{flat}}$. This is because this quantity is purely geometrical, and so will not evolve while the perturbation is beyond the horizon. The two-point correlation function of the comoving curvature perturbation is:

$$\langle\tilde{\psi}_{\text{com}}(\vec{k})\tilde{\psi}_{\text{com}}(-\vec{k})\rangle = \frac{1}{\beta(\varphi)^2}\langle\tilde{\chi}_{\text{flat}}(\vec{k})\tilde{\chi}_{\text{flat}}(-\vec{k})\rangle \tag{5.17}$$

A conventional way to express the two-point correlator of a scalar field in cosmology is to introduce the power-spectrum:

$$P_s(k) = \frac{k^3}{2\pi^2}\,\langle\tilde{\psi}_{\text{com}}(\vec{k})\,\tilde{\psi}_{\text{com}}(-\vec{k})\rangle\,. \tag{5.18}$$

[10]The comoving curvature perturbation is sometimes denoted by either ζ or $\mathcal{R}$.

Collecting formulae, our result for the power spectrum of the comoving curvature perturbation becomes:

$$\begin{aligned} P_s(k) &= \left(\frac{H(\varphi)}{2\pi}\right)^2 \frac{1}{\beta(\varphi)^2} \mathcal{A}_s\left(\bar{\varphi}\left(\frac{k}{a},\varphi\right)\right) \times \\ &\times \exp\left(-\int_{aM}^{k} d\log\left(\frac{k'}{aM}\right)\left(\beta\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)^2 + 2\gamma\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)\right)\right). \end{aligned} \tag{5.19}$$

This formula can be written more concisely as

$$P_s(k) = \left(\frac{H(\bar{\varphi}(k/a,\varphi))}{2\pi}\right)^2 \frac{1}{\beta(\bar{\varphi}(k/a,\varphi))^2} \mathcal{A}_s\left(\bar{\varphi}\left(\frac{k}{a},\varphi\right)\right) \tag{5.20}$$

but the form (5.19) is more useful because it makes the k dependence explicit.

It is instructive to consider how our result (5.19) reproduces the familiar results from slow-roll inflation. First we rewrite the RG-parameters β and γ in terms of the usual first order slow-roll parameters $\bar{\epsilon}$ and $\bar{\eta}$:

$$\beta(\varphi)^2 = 2\,\bar{\epsilon} = 4\left(\frac{\partial_\varphi H}{H}\right)^2 \tag{5.21}$$

$$\gamma(\varphi) = \bar{\epsilon} - \bar{\eta} = 2\left(\frac{\partial_\varphi H}{H}\right)^2 - 2\,\frac{\partial_\varphi^2 H}{H}. \tag{5.22}$$

The leading order in the slow-roll expansion amounts to assuming that the $\bar{\epsilon}$ and $\bar{\eta}$ are constants. Additionally, we can take the amplitude $\mathcal{A}_s = 1$. The expression (5.19) then gives the power spectrum:

$$P_s(k) = \left(\frac{H}{2\pi}\right)^2 \frac{1}{2\,\bar{\epsilon}} \left(\frac{k}{aM}\right)^{2\bar{\eta}-4\bar{\epsilon}}. \tag{5.23}$$

The explicit dependence on the reference scale M compensates the implicit dependence of H, $\bar{\epsilon}$, and $\bar{\eta}$ on the value of φ. Although M is arbitrary, in principle, it is natural to take it of order the horizon scale $M = H$ and, correspondingly, evaluate the other parameters at horizon crossing.

A standard way to present the result is to introduce the spectral index, defined by fitting the momentum dependence to a power law:

$$P_s(k) \propto \left(\frac{k}{aH}\right)^{n_s-1} \tag{5.24}$$

Keeping only terms that are first order in the slow-roll parameters gives:

$$n_s = 1 - 4\,\bar{\epsilon} + 2\,\bar{\eta} \tag{5.25}$$

which is the standard first order result for slow-roll inflation.

The power spectrum (5.23) depends on the slow-roll parameters in the exponent. Since these are small one might expand

$$\left(\frac{k}{aM}\right)^{2\bar{\eta}-4\bar{\epsilon}} \simeq 1 + (2\bar{\eta} - 4\bar{\epsilon})\log\left(\frac{k}{aM}\right) + \cdots \tag{5.26}$$

where, according to the slow-roll expansion, higher orders are negligible. The problem with this is that $(2\bar{\eta} - 4\bar{\epsilon}) \log\left(\frac{k}{a\,M}\right)$ can be large even when $2\bar{\eta} - 4\bar{\epsilon}$ is not. The content of the RG-improved slow-roll expansion is that the power spectrum (5.23) is in fact accurate to *all* orders in the (potentially large) logarithms $\bar{\eta}\log k$ and $\bar{\epsilon}\log k$, albeit only to the leading order in $\bar{\eta}, \bar{\epsilon}$.

It would be interesting to apply the RG-improved pertrubation theory to higher orders in the slow-roll expansion. Amplitudes equivalent to our $\mathcal{A}_s$ were computed to the second order by Stewart and Lyth [61] (and reproduced by us in [7]), and to the third order by Gong and Stewart [62, 63, 64]. Inserting such standard results in (5.19) or (5.20) the RG-improved power spectrum follows. Since the exploration of explicit models of inflation is somewhat outside the main line of development in this paper we defer such a study to a separate publication [65]. For the present let us simply note that, for some inflationary potentials, the improved higher order results can differ substantially from the standard ones.

5.2 The tensor modes

Up to this point we have neglected contributions due to the tensor modes, which are completely decoupled from the scalar fluctuation at quadratic order. We will now repeat the analysis with the tensor modes included. In the analogy with a three dimensional euclidean field theory, the starting point is the full partition function, with the tensor modes included. The couplings between sources and operators are of the form:

$$\int d^3x \left[\frac{1}{2}\, h_{ij}\, T^{ij} + \chi\, \mathcal{O}\right] = \int d^3x \left[\frac{1}{2}\, \phi_{ij}\, T^i_j + \chi_f\, \mathcal{O}\right]. \tag{5.27}$$

For the scalars we used the conformal Ward identity (4.40) to rewrite $\psi T^i_i + \chi\mathcal{O}$ as the gauge invariant quantity $\chi_f\mathcal{O}$. If we choose a basis $e^{\pm}_{ij}$ for the tensor modes:

$$\phi_{ij} = 2\left(u_+\, e^+_{ij} + u_-\, e^-_{ij}\right) \tag{5.28}$$

with $e^{\pm}_{ij}e^{\pm}_{ij} = 1$ and $e^{\pm}_{ij}e^{\mp}_{ij} = 0$, then the two polarization states of the tensor appear in the action (3.31) as minimally coupled, massless scalars with a canonical kinetic term. In the partition function they serve as sources for operators t_+ and t_-, corresponding to the two components of the traceless, divergence-free part of the field theory stress tensor:

$$Z[\varphi(\vec{x}), \tilde{g}_{ij}(\vec{x})] = \left\langle \exp\left(\int d^3x \left[u_+(\vec{x})t_+(\vec{x}) + u_-(\vec{x})t_-(\vec{x}) + \chi_f(\vec{x})\, \mathcal{O}(\vec{x})\right]\right)\right\rangle \tag{5.29}$$

To quadratic order we can compute the gaussian integral as in (5.4) and relate the two-point function for each mode $u_\pm$ to the correlator of the corresponding operator $t_\pm$:

$$\langle \tilde{u}_+(\vec{k})\tilde{u}_+(-\vec{k})\rangle = \frac{1}{2\,\langle \tilde{t}_+(\vec{k})\tilde{t}_+(-\vec{k})\rangle}\,. \tag{5.30}$$

The Callan-Symanzik equation for correlators of the tensor modes is derived from (4.13) by varying with respect to the tensor sources. Since the β-function is independent of the gravitational field — it depends on the scalar field only — the variations commute with all terms in the equation. The Callan-Symanzik equation of the tensor modes is therefore simply:

$$\left(a\,\frac{\partial}{\partial\,a}+\beta(\varphi)\,\frac{\partial}{\partial\,\varphi}\right)\,\langle t_\pm(\vec{k})\,t_\pm(-\vec{k})\rangle=0\,. \tag{5.31}$$

with no term corresponding to an anomalous dimension. It is simple to integrate this equation by repeating the steps that lead to (5.16) for the correlator of scalar fields, with the simplification that now the anomalous dimension $\gamma_t = 0$. The correlator of the tensor fields $\tilde{u}_\pm$ thus takes the form:

$$\langle\tilde{u}_\pm(\vec{k})\,\tilde{u}_\pm(-\vec{k})\rangle=\frac{H(\varphi)^2}{2\,k^3}\mathcal{A}_t\left(\bar{\varphi}\left(\frac{k}{a},\varphi\right)\right)\exp\left(-\int_{aM}^{k}d\log\left(\frac{k'}{aM}\right)\beta\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)^2\right) \tag{5.32}$$

The power spectrum for the tensor modes is introduced as:

$$P_t(\vec{k}) = \frac{k^3}{2\pi^2}\,\langle\tilde{\phi}_{ij}(\vec{k})\,\tilde{\phi}_{ij}(-\vec{k})\rangle \tag{5.33}$$

$$= \frac{k^3}{2\pi^2}\,8\,\langle\tilde{u}_+(\vec{k})\,\tilde{u}_+(-\vec{k})\rangle\,. \tag{5.34}$$

Factors of two appear in the second line from the decomposition (5.28) and also because we have written the result entirely in terms of $\langle u_+u_+\rangle$; the u_+ and u_- correlators are identical in the absence of polarizing sources. Our result for the RG improved power spectrum of the tensor modes becomes:

$$P_t(\vec{k})=\frac{H(\varphi)^2}{(2\pi)^2}\;8\mathcal{A}_t\left(\bar{\varphi}\left(\frac{k}{a},\varphi\right)\right)\exp\left(-\int_{aM}^{k}d\log(\frac{k'}{aM})\,\beta\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)^2\right) \tag{5.35}$$

As for scalars, the amplitude $\mathcal{A}_t$ is not determined by symmetries alone. However, in slowroll inflation, $\mathcal{A}_t = 1$ well after horizon crossing. Our expression then gives the full dependence on the scale k, and it acts as a resummation of the corrections to slow roll which, in general, can be large.

The spectral index n_t for the tensor modes is defined by fitting the power spectrum to the form:

$$P_t(\vec{k})\propto\left(\frac{k}{aH}\right)^{n_t} \tag{5.36}$$

Note that the convention for the tensor index n_t is shifted by one in comparison with the scaler index (5.24); so scale invariance corresponds to $n_t = 0$ for tensor modes. Repeating the computation of the spectral index for scalar modes (5.25) we find:

$$n_t = k\frac{\partial}{\partial\,k}\,\log P_t(\vec{k}) = -\beta(\varphi)^2 \tag{5.37}$$

for the spectral index of the tensor modes. Our result agrees with the standard result $n_t = -2\epsilon$, when it is expressed in in terms of the slow-roll parameter ϵ (5.21).

The ratio of the tensor and power spectra is:

$$\frac{P_t}{P_s}=8\beta(\varphi)^2\frac{\mathcal{A}_t(\bar{\varphi}(k'/a,\varphi))}{\mathcal{A}_s(\bar{\varphi}(k'/a,\varphi))}\exp\left(\int_{aM}^{k}d\log\left(\frac{k'}{aM}\right)2\gamma\left(\bar{\varphi}\left(\frac{k'}{a},\varphi\right)\right)\right) \tag{5.38}$$

In slow roll inflation where $\mathcal{A}_t = \mathcal{A}_s = 1$ and the exponential is negligible this reduces to the famous consistency condition:

$$\frac{P_t}{P_s} = 8\beta(\varphi)^2 = -8n_t \tag{5.39}$$

on the physical observables. Our modest elaboration of this standard result is that our result incorporates the evolution of this ratio with scale k.

6. Outlook

Before concluding this paper we would like to briefly mention several applications and extensions of our results which we think are worth pursuing further:

- A concrete result of the investigation in this paper are the explicit, gauge invariant actions (3.31), (3.34), and (3.35) for fluctuations in spacetimes with a boundary. There are numerous applications of actions such as these in the context of the AdS/CFT correspondence generally, and specificaly in the context of warped brane-world models, such as those of Randall-Sundrum type. Moreover, the nontrivial role of counterterms in implementing diffeomorphism invariance, and so obtaining gauge invariant actions, might well resolve various puzzles in those other settings.

- We have introduced our concept of RG improved computations of CMB spectra, based on integrating the constraints from invariance under infinitesimal diffeomorphisms, as expressed by the Callan-Symanzik equation. It would be interesting to make this procedure explicit for concrete potentials of interest in inflationary cosmology, and so identify cases where these corrections are in fact significant.

- It would be interesting to extend our analysis beyond quadratic order, and obtain constraints on correlators of more fields than two. The main difficulty in carrying this out is that one must implement diffeomorphism invariance, or gauge invariance, beyond the leading order. This is nontrivial, although the hamiltonian formalism should automatize a significant part of the work.

- The cancellation of divergent terms in the on-shell action by a finite number of boundary counterterms is dependent on some constraints on the backgrounds: it is known to work for asymptotically (A)dS-spacetimes, and we show that the mechanism extends to large classes of FRW cosmologies. The situation in more general spacetimes, including asymptotically flat ones, is less clear. It would be interesting to explore whether our method is useful for adapting the boundary counterterm method to a wider class of spacetimes. Additionally, the method may serve as a basis for constraining the boundary terms which make the notion of quasi-local mass and energy in gravitational theories inherently ambiguous [57, 26, 54].

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References

[1] J.B. Hartle and S.W. Hawking, *Wave function of the universe*, *Phys. Rev.* **D 28** (1983) 2960.

[2] D.S. Salopek and J.R. Bond, *Nonlinear evolution of long wavelength metric fluctuations in inflationary models*, *Phys. Rev.* **D 42** (1990) 3936.

[3] D.S. Salopek and J.M. Stewart, *Hypersurface invariant approach to cosmological perturbations*, *Phys. Rev.* **D 51** (1995) 517 [astro-ph/9409055].

[4] D.S. Salopek, *Coordinate-free solutions for cosmological superspace*, *Phys. Rev.* **D 56** (1997) 2057 [astro-ph/9801161].

[5] J. Parry, D.S. Salopek and J.M. Stewart, *Solving the Hamilton-Jacobi equation for general relativity*, *Phys. Rev.* **D 49** (1994) 2872 [gr-qc/9310020].

[6] J. Maldacena, *Non-gaussian features of primordial fluctuations in single field inflationary models*, *J. High Energy Phys.* **05** (2003) 013 [astro-ph/0210603].

[7] F. Larsen and R. McNees, *Inflation and de Sitter holography*, *J. High Energy Phys.* **07** (2003) 051 [hep-th/0307026].

[8] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[9] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[10] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111].

[11] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[12] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, *Commun. Math. Phys.* **208** (1999) 413 [hep-th/9902121].

[13] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev.* **D 60** (1999) 104001 [hep-th/9903238].

[14] P. Kraus, F. Larsen and R. Siebelink, *The gravitational action in asymptotically AdS and flat spacetimes*, *Nucl. Phys.* **B 563** (1999) 259 [hep-th/9906127].

[15] E.M. Lifshitz, I.M. Khalatnikov, *Problems of relativistic cosmology*, *Uspekhi Phys. Nauk.* **80** (1963) 391, (*Translation in Soviet Physics Uspekhi*, **6** (1964) 495).

[16] J.M. Bardeen, *Gauge invariant cosmological perturbations*, *Phys. Rev.* **D 22** (1980) 1882.

[17] V.F. Mukhanov, *Quantum theory of gauge invariant cosmological perturbations*, *Sov. Phys. JETP* **67** (1988) 1297.

[18] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, *Theory of cosmological perturbations, part 1. Classical perturbations, part 2. Quantum theory of perturbations, part 3. Extensions*, *Phys. Rev.* **215** (1992) 203.

[19] A. Riotto, *Inflation and the theory of cosmological perturbations*, hep-ph/0210162.

[20] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Reading, Addison-Wesley 1995.

[21] A. Strominger, *The dS/CFT correspondence*, *J. High Energy Phys.* **10** (2001) 034 [hep-th/0106113].

[22] A. Strominger, *Inflation and the dS/CFT correspondence*, *J. High Energy Phys.* **11** (2001) 049 [hep-th/0110087].

[23] D. Klemm, *Some aspects of the de Sitter/CFT correspondence*, *Nucl. Phys.* **B 625** (2002) 295 [hep-th/0106247].

[24] M. Spradlin and A. Volovich, *Vacuum states and the S-matrix in dS/CFT*, *Phys. Rev.* **D 65** (2002) 104037 [hep-th/0112223].

[25] E. Witten, *Quantum gravity in de Sitter space*, hep-th/0106109.

[26] V. Balasubramanian, J. de Boer and D. Minic, *Exploring de Sitter space and holography*, *Class. and Quant. Grav.* **19** (2002) 5655 [hep-th/0207245].

[27] F. Larsen, J.P. van der Schaar and R.G. Leigh, *De Sitter holography and the cosmic microwave background*, *J. High Energy Phys.* **04** (2002) 047 [hep-th/0202127].

[28] J.P. van der Schaar, *Inflationary perturbations from deformed CFT*, *J. High Energy Phys.* **01** (2004) 070 [hep-th/0307271].

[29] K. Schalm, G. Shiu and J.P. van der Schaar, *Decoupling in an expanding universe: boundary RG-flow affects initial conditions for inflation*, *J. High Energy Phys.* **04** (2004) 076 [hep-th/0401164].

[30] D.H. Lyth and A. Riotto, *Particle physics models of inflation and the cosmological density perturbation*, *Phys. Rept.* **314** (1999) 1 [hep-ph/9807278].

[31] J.A. Peacock, *Cosmological physics*, Cambridge University Press, Cambridge 1999.

[32] P.J. Peebles, *Principles of physical cosmology*, Princeton University Press, Princeton 1993.

[33] A.R. Liddle and D.H. Lyth, *Cosmological inflation and large-scale structure*, Cambridge University Press, Cambridge 2000.

[34] G.W. Gibbons and S.W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev.* **D 15** (1977) 2752.

[35] A.A. Starobinsky, *Spectrum of relict gravitational radiation and the early state of the universe*, *Sov. Phys. JETP Lett.* **30** (1979) 682.

[36] A.A. Starobinsky, *Dynamics of phase transition in the new inflationary universe scenario and generation of perturbations*, *Phys. Lett.* **B 117** (1982) 175.

[37] A.A. Starobinsky, *The perturbation spectrum evolving from a nonsingular, initially de Sitter cosmology, and the microwave background anisotropy*, *Sov. Astron. Lett.* **9** (1983) 302.

[38] V.F. Mukhanov and G.V. Chibisov, *Quantum fluctuation and 'nonsingular' universe*, *Sov. Phys. JETP Lett.* **33** (1981) 532.

[39] V.F. Mukhanov and G.V. Chibisov, *The vacuum energy and large scale structure of the universe*, *Sov. Phys. JETP* **56** (1982) 258.

[40] A.H. Guth and S.Y. Pi, *Fluctuations in the new inflationary universe*, *Phys. Rev. Lett.* **49** (1982) 1110.

[41] S.W. Hawking, *The development of irregularities in a single bubble inflationary universe*, *Phys. Lett.* **B 115** (1982) 295.

[42] J.M. Bardeen, P.J. Steinhardt and M.S. Turner, *Spontaneous creation of almost scale - free density perturbations in an inflationary universe*, *Phys. Rev.* **D 28** (1983) 679.

[43] C.L. Bennett et al., *First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: preliminary maps and basic results*, *Astrophys. J. Suppl.* **148** (2003) 1 [astro-ph/0302207].

[44] G. Hinshaw et al., *First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: angular power spectrum*, *Astrophys. J. Suppl.* **148** (2003) 135 [astro-ph/0302217].

[45] H.V. Peiris et al., *First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: implications for inflation*, *Astrophys. J. Suppl.* **148** (2003) 213 [astro-ph/0302225].

[46] J. de Boer, E. Verlinde and H. Verlinde, *On the holographic renormalization group*, *J. High Energy Phys.* **08** (2000) 003 [hep-th/9912012].

[47] M. Bianchi, D.Z. Freedman and K. Skenderis, *Holographic renormalization*, *Nucl. Phys.* **B 631** (2002) 159 [hep-th/0112119].

[48] M. Bianchi, D.Z. Freedman and K. Skenderis, *How to go with an RG flow*, *J. High Energy Phys.* **08** (2001) 041 [hep-th/0105276].

[49] O. DeWolfe and D.Z. Freedman, *Notes on fluctuations and correlation functions in holographic renormalization group flows*, hep-th/0002226.

[50] V. Balasubramanian and P. Kraus, *Spacetime and the holographic renormalization group*, *Phys. Rev. Lett.* **83** (1999) 3605 [hep-th/9903190].

[51] D.Z. Freedman, S.S. Gubser, K. Pilch and N.P. Warner, *Renormalization group flows from holography supersymmetry and a c-theorem*, *Adv. Theor. Math. Phys.* **3** (1999) 363 [hep-th/9904017].

[52] S. Nojiri and S.D. Odintsov, *Asymptotically de Sitter dilatonic space-time, holographic rg flow and conformal anomaly from (dilatonic) dS/CFT correspondence*, *Phys. Lett.* **B 531** (2002) 143 [hep-th/0201210].

[53] U.H. Danielsson, *Inflation, holography and the choice of vacuum in de Sitter space*, *J. High Energy Phys.* **07** (2002) 040 [hep-th/0205227].

[54] V. Balasubramanian, J. de Boer and D. Minic, *Mass, entropy and holography in asymptotically de Sitter spaces*, *Phys. Rev.* **D 65** (2002) 123508 [hep-th/0110108].

[55] W.H. Kinney, *Inflation: flow, fixed points and observables to arbitrary order in slow roll*, *Phys. Rev.* **D 66** (2002) 083508 [astro-ph/0206032].

[56] W.H. Kinney, E.W. Kolb, A. Melchiorri and A. Riotto, *Wmapping inflationary physics*, *Phys. Rev.* **D 69** (2004) 103516 [hep-ph/0305130].

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[57] J.D. Brown, S.R. Lau and J.W. York, *Action and energy of the gravitational field*, gr-qc/0010024.

[58] B. McInnes, *String theory and the shape of the universe*, hep-th/0401035.

[59] B. McInnes, *Orbifold physics and de Sitter spacetime*, *Nucl. Phys.* **B 692** (2004) 270 [hep-th/0311055].

[60] A. Guijosa and D.A. Lowe, *A new twist on dS/CFT*, *Phys. Rev.* **D 69** (2004) 106008 [hep-th/0312282].

[61] E.D. Stewart and D.H. Lyth, *A more accurate analytic calculation of the spectrum of cosmological perturbations produced during inflation*, *Phys. Lett.* **B 302** (1993) 171 [gr-qc/9302019].

[62] J.-O. Gong and E.D. Stewart, *The density perturbation power spectrum to second-order corrections in the slow-roll expansion*, *Phys. Lett.* **B 510** (2001) 1 [astro-ph/0101225].

[63] J.-O. Gong and E.D. Stewart, *The power spectrum for a multi-component inflaton to second-order corrections in the slow-roll expansion*, *Phys. Lett.* **B 538** (2002) 213 [astro-ph/0202098].

[64] J. Choe, J.-O. Gong and E.D. Stewart, *Second order general slow-roll power spectrum*, *JCAP* **07** (2004) 012 [hep-ph/0405155].

[65] F. Larsen and R. McNees, work in progress.

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